Rotationally invariant viscoelastic medium with a non-symmetric stiffness matrix

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ABSTRACT

The stiffness matrix of a viscoelastic medium is symmetric in the low–frequency and high–frequency limits, but not for finite frequencies. We thus consider a non–symmetric stiffness matrix in this paper. We determine the general form of a rotationally invariant non–symmetric stiffness matrix of a viscoelastic medium. It is described by three additional complex–valued parameters in comparison with a rotationally invariant symmetric stiffness matrix of a transversely isotropic (uniaxial) viscoelastic medium with a symmetric stiffness matrix. As a consequence, we find that the stiffness matrix of an isotropic viscoelastic medium is always symmetric.

Keywords: viscoelastic media, stiffness tensor, elastic moduli, transverse isotropy, symmetry axis

1. INTRODUCTION

The 3×3×3×3 complex–valued frequency–domain stiffness tensor (elastic tensor, tensor of elastic moduli) $c^{ijkl} = c^{ijkl}(x^m, \omega)$, projecting the strain tensor onto the stress tensor, is symmetric with respect to the first pair of indices

$$c^{ijkl} = c^{jikl} \tag{1}$$

and with respect to the second pair of indices

$$c^{ijkl} = c^{ijlk} \quad . \tag{2}$$

It is thus frequently expressed in the form of the 6×6 stiffness matrix whose lines correspond to the first pair of indices and columns to the second pair of indices according to the *Voigt (1910)* notation.

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices,

$$c^{ijkl} = c^{klij} \quad . \tag{3}$$

The 6×6 stiffness matrix is thus symmetric in an elastic medium.

However, the above mentioned proof does not apply to a viscoelastic medium, and we do not know whether symmetry (3) holds in a viscoelastic medium. In a viscoelastic medium, symmetry (3) was proved in the low–frequency and high–frequency limits only (*Gurtin and Herrera, 1965, theorems 3.1 and 3.2*; *Christensen, 1971, Eqs 3.39–3.40*; *Fabrizio and Morro, 1988, corollaries 1 and 2; 1992, Eqs 3.28–3.29*; *Carcione, 2015, Eq. 2.24*), but not for finite frequencies (*Rogers and Pipkin, 1963*).

Analogously to *de Hoop (1995)*, *Thomson (1997)* and *Klimeš (2018b; 2021)*, we thus consider a *non–symmetric stiffness matrix*,

$$c^{ijkl} \neq c^{klij} \quad , \tag{4}$$

in this paper. Note that the material stability conditions (*Fabrizio and Morro, 1992, Eqs 4.3.8–4.3.9*) require the real part of frequency–domain stiffness matrix (4) to be positive–semidefinite, and the imaginary part to be negative–semidefinite for the Fourier transform of the linear constitutive equation with the sign convention according to *Červený (2001, Eq. A.1.2)*.

We suppose that a viscoelastic medium with a non–symmetric stiffness matrix is invariant with respect to the rotation about a given symmetry axis, and determine the general form of the stiffness matrix in Sections 2–5. We choose the third coordinate axis in the direction of the axis of rotation in Sections 2–4 without a loss of generality. We then transform the derived stiffness matrix into general coordinates in Section 5. Although this medium is transversely isotropic (uniaxial), we prefer to refer to it as rotationally invariant because many readers may have a transversely isotropic (uniaxial) medium connected with a symmetric stiffness matrix.

We calculate the derivative of the stiffness tensor with respect to the angle of rotation in Section 2. We put the derivative equal to zero, and obtain the system of equations for the elements of the stiffness tensor, analogous to the equations of *Cowin and Mehrabadi (1987)*.

We express these equations in the coordinate system attached to the symmetry axis in Section 3, and solve them in Section 4. We then determine the general form of a rotationally invariant non–symmetric stiffness matrix of a viscoelastic medium in general coordinates in Section 5.

There is also another very interesting related problem. It is well known that some liquids like syrups exhibit the optical activity. Can an isotropic viscoelastic medium exhibit an analogous "acoustical activity"? This question is answered in Section 6. I admit that the presented study was originally initiated by my curiousness about a possibility of such an acoustical activity.

We assume Cartesian coordinates with the unit metric tensor. The lower–case Roman indices take values 1, 2 and 3. The Einstein summation over repetitive lower–case Roman indices is used throughout the paper.

2. DERIVATIVE OF THE STIFFNESS TENSOR WITH RESPECT TO THE ANGLE OF ROTATION

We denote the stiffness tensor c^{ijkl} or the density–normalized stiffness tensor c^{ijkl}/ϱ of a viscoelastic medium by a^{ijkl}. We denote the unit symmetry vector in the direction of the symmetry axis by t_i.

Transformation matrix $R_{in}(\varphi, t_a)$ corresponding to the rotation of vectors about a given unit vector t_a by angle φ is an orthogonal matrix, with $R_{in}(0, t_a) = \delta_{in}$, where Kronecker delta δ_{in} represents the elements of the identity matrix, see *Klimeš (2018a, Eq. 4)*. The derivative of the transformation matrix at $\varphi = 0$ reads

$$\frac{\mathrm{d}R_{in}}{\mathrm{d}\varphi}(0, t_a) = -S_{in} \quad , \tag{5}$$

where

$$S_{in} = \varepsilon_{inr} t_r \tag{6}$$

(*Klimeš, 2018a, Eq. 3*). Here ε_{ijk} is the Levi–Civita symbol.

The rotated stiffness tensor reads (*Klimeš, 2016, Eq. 2*)

$$a^{ijkl}(\varphi, t_a) = R_{ip}(\varphi, t_a)\, R_{jq}(\varphi, t_b)\, R_{kr}(\varphi, t_c)\, R_{ls}(\varphi, t_d)\, a^{pqrs} \quad , \tag{7}$$

where a^{pqrs} without arguments is the non–rotated tensor. Note that tensor transformation (7) may numerically be implemented in terms of the matrix *Bond (1943)* transformation which is computationally efficient but impractical in theory.

The derivative $\frac{\mathrm{d}a^{ijkl}}{\mathrm{d}\varphi}(0, t_a)$ of stiffness tensor $a^{ijkl}(\varphi, t_a)$ with respect to the angle φ of rotation at $\varphi = 0$ follows directly from transformation (7) with derivative (5) (*Klimeš, 2016, Eq. 3*),

$$\frac{\mathrm{d}a^{ijkl}}{\mathrm{d}\varphi}(0, t_a) = -S_{in} a^{njkl} - S_{jn} a^{inkl} - S_{kn} a^{ijnl} - S_{ln} a^{ijkn} \quad . \tag{8}$$

We put $\frac{\mathrm{d}a^{ijkl}}{\mathrm{d}\varphi}(0, t_a) = 0$, and obtain the system of equations

$$S_{in} a^{njkl} + S_{jn} a^{inkl} + S_{kn} a^{ijnl} + S_{ln} a^{ijkn} = 0 \tag{9}$$

for the rotationally invariant non–symmetric stiffness tensor of a viscoelastic medium, equivalent to the equations of *Cowin and Mehrabadi (1987)*.

3. COORDINATE SYSTEM ATTACHED TO THE SYMMETRY AXIS

We choose the coordinate system where the third coordinate axis coincides with the symmetry axis. In this coordinate system, the symmetry vector reads

$$t_a = (0, 0, 1) \quad . \tag{10}$$

Matrix (6) then takes form

$$S_{ia} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{11}$$

We express the stiffness tensor in the *Voigt (1910)* notation as

$$a^{ijkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 11 & 22 & 33 & 23 & 13 & 12 \end{matrix} \\ \begin{pmatrix} a^{1111} & a^{1122} & a^{1133} & a^{1123} & a^{1113} & a^{1112} \\ a^{2211} & a^{2222} & a^{2233} & a^{2223} & a^{2213} & a^{2212} \\ a^{3311} & a^{3322} & a^{3333} & a^{3323} & a^{3313} & a^{3312} \\ a^{2311} & a^{2322} & a^{2333} & a^{2323} & a^{2313} & a^{2312} \\ a^{1311} & a^{1322} & a^{1333} & a^{1323} & a^{1313} & a^{1312} \\ a^{1211} & a^{1222} & a^{1233} & a^{1223} & a^{1213} & a^{1212} \end{pmatrix} \end{array} . \tag{12}$$

The individual addends on the right–hand side of relation (8) in the *Voigt (1910)* notation then read

$$S_{ia}\, a^{ajkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 11 & 22 & 33 & 23 & 13 & 12 \end{matrix} \\ \begin{pmatrix} a^{1211} & a^{1222} & a^{1233} & a^{1223} & a^{1213} & a^{1212} \\ -a^{1211} & -a^{1222} & -a^{1233} & -a^{1223} & -a^{1213} & -a^{1212} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -a^{1311} & -a^{1322} & -a^{1333} & -a^{1323} & -a^{1313} & -a^{1312} \\ a^{2311} & a^{2322} & a^{2333} & a^{2323} & a^{2313} & a^{2312} \\ a^{2211} & a^{2222} & a^{2233} & a^{2223} & a^{2213} & a^{2212} \end{pmatrix} \end{array} , \tag{13}$$

$$S_{jb}\, a^{ibkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 11 & 22 & 33 & 23 & 13 & 12 \end{matrix} \\ \begin{pmatrix} a^{1211} & a^{1222} & a^{1233} & a^{1223} & a^{1213} & a^{1212} \\ -a^{1211} & -a^{1222} & -a^{1233} & -a^{1223} & -a^{1213} & -a^{1212} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -a^{1111} & -a^{1122} & -a^{1133} & -a^{1123} & -a^{1113} & -a^{1112} \end{pmatrix} \end{array} , \tag{14}$$

$$S_{kc}\, a^{ijcl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 11 & 22 & 33 & 23 & 13 & 12 \end{matrix} \\ \begin{pmatrix} a^{1112} & -a^{1112} & 0 & -a^{1113} & a^{1123} & a^{1122} \\ a^{2212} & -a^{2212} & 0 & -a^{2213} & a^{2223} & a^{2222} \\ a^{3312} & -a^{3312} & 0 & -a^{3313} & a^{3323} & a^{3322} \\ a^{2312} & -a^{2312} & 0 & -a^{2313} & a^{2323} & a^{2322} \\ a^{1312} & -a^{1312} & 0 & -a^{1313} & a^{1323} & a^{1322} \\ a^{1212} & -a^{1212} & 0 & -a^{1213} & a^{1223} & a^{1222} \end{pmatrix} \end{array} \tag{15}$$

and

$$S_{ld}\, a^{ijkd} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{matrix} 11 & 22 & 33 & 23 & 13 & 12 \end{matrix} \\ \begin{pmatrix} a^{1112} & -a^{1112} & 0 & 0 & 0 & -a^{1111} \\ a^{2212} & -a^{2212} & 0 & 0 & 0 & -a^{2211} \\ a^{3312} & -a^{3312} & 0 & 0 & 0 & -a^{3311} \\ a^{2312} & -a^{2312} & 0 & 0 & 0 & -a^{2311} \\ a^{1312} & -a^{1312} & 0 & 0 & 0 & -a^{1311} \\ a^{1212} & -a^{1212} & 0 & 0 & 0 & -a^{1211} \end{pmatrix} \end{array} . \tag{16}$$

We can now solve matrix equation (9) with matrices (13)–(16) in order to obtain rotationally invariant stiffness matrix (12) in this coordinate system attached to the symmetry axis.

4. ROTATIONALLY INVARIANT NON–SYMMETRIC STIFFNESS MATRIX IN THE COORDINATE SYSTEM ATTACHED TO THE SYMMETRY AXIS

The first row of matrix equation (9) with addends (13)–(16) yields equations

$$a^{1211} + a^{1211} + a^{1112} + a^{1112} = 0 \quad , \tag{17.11}$$

$$a^{1222} + a^{1222} - a^{1112} - a^{1112} = 0 \quad , \tag{17.12}$$

$$a^{1233} + a^{1233} = 0 \quad , \tag{17.13}$$

$$a^{1223} + a^{1223} - a^{1113} = 0 \quad , \tag{17.14}$$

$$a^{1213} + a^{1213} + a^{1123} = 0 \quad , \tag{17.15}$$

$$a^{1212} + a^{1212} + a^{1122} - a^{1111} = 0 \quad . \tag{17.16}$$

The second row of matrix equation (9) with addends (13)–(16) yields equations

$$-a^{1211} - a^{1211} + a^{2212} + a^{2212} = 0 \quad , \tag{17.21}$$

$$-a^{1222} - a^{1222} - a^{2212} - a^{2212} = 0 \quad , \tag{17.22}$$

$$-a^{1233} - a^{1233} = 0 \quad , \tag{17.23}$$

$$-a^{1223} - a^{1223} - a^{2213} = 0 \quad , \tag{17.24}$$

$$-a^{1213} - a^{1213} + a^{2223} = 0 \quad , \tag{17.25}$$

$$-a^{1212} - a^{1212} + a^{2222} - a^{2211} = 0 \quad . \tag{17.26}$$

The third row of matrix equation (9) with addends (13)–(16) yields equations

$$a^{3312} + a^{3312} = 0 \quad , \tag{17.31}$$

$$-a^{3312} - a^{3312} = 0 \quad , \tag{17.32}$$

$$0 = 0 \quad , \tag{17.33}$$

$$-a^{3313} = 0 \quad , \tag{17.34}$$

$$a^{3323} = 0 \quad , \tag{17.35}$$

$$a^{3322} - a^{3311} = 0 \quad . \tag{17.36}$$

The fourth row of matrix equation (9) with addends (13)–(16) yields equations

$$-a^{1311} + a^{2312} + a^{2312} = 0 \quad , \tag{17.41}$$

$$-a^{1322} - a^{2312} - a^{2312} = 0 \quad , \tag{17.42}$$

$$-a^{1333} = 0 \quad , \tag{17.43}$$

$$-a^{1323} - a^{2313} = 0 \quad , \tag{17.44}$$

$$-a^{1313} + a^{2323} = 0 \quad , \tag{17.45}$$

$$-a^{1312} + a^{2322} - a^{2311} = 0 \quad . \tag{17.46}$$

The fifth row of matrix equation (9) with addends (13)–(16) yields equations

$$a^{2311} + a^{1312} + a^{1312} = 0 \quad , \tag{17.51}$$

$$a^{2322} - a^{1312} - a^{1312} = 0 \quad , \tag{17.52}$$

$$a^{2333} = 0 \quad , \tag{17.53}$$
$$a^{2323} - a^{1313} = 0 \quad , \tag{17.54}$$
$$a^{2313} + a^{1323} = 0 \quad , \tag{17.55}$$
$$a^{2312} + a^{1322} - a^{1311} = 0 \quad . \tag{17.56}$$

The sixth row of matrix equation (9) with addends (13)–(16) yields equations

$$a^{2211} - a^{1111} + a^{1212} + a^{1212} = 0 \quad , \tag{17.61}$$
$$a^{2222} - a^{1122} - a^{1212} - a^{1212} = 0 \quad , \tag{17.62}$$
$$a^{2233} - a^{1133} = 0 \quad , \tag{17.63}$$
$$a^{2223} - a^{1123} - a^{1213} = 0 \quad , \tag{17.64}$$
$$a^{2213} - a^{1113} + a^{1223} = 0 \quad , \tag{17.65}$$
$$a^{2212} - a^{1112} + a^{1222} - a^{1211} = 0 \quad . \tag{17.66}$$

We now solve the system of equations (17).

Equation (17.33) contains no information. We use ≡ to denote equivalent equations. Eight equations (17.13)≡(17.23), (17.31)≡(17.32), (17.34), (17.35), (17.43) and (17.53) read

$$a^{1233} = 0 \quad , \qquad a^{3312} = 0 \quad , \qquad a^{3313} = 0 \quad ,$$
$$a^{3323} = 0 \quad , \qquad a^{1333} = 0 \quad , \qquad a^{2333} = 0 \quad . \tag{18}$$

Equations (17.14), (17.24) and (17.65) yield

$$a^{1113} = 0 \quad , \qquad a^{2213} = 0 \quad , \qquad a^{1223} = 0 \quad . \tag{19}$$

Equations (17.15), (17.25) and (17.64) yield

$$a^{1123} = 0 \quad , \qquad a^{2223} = 0 \quad , \qquad a^{1213} = 0 \quad . \tag{20}$$

Equations (17.41), (17.42) and (17.56) yield

$$a^{1311} = 0 \quad , \qquad a^{1322} = 0 \quad , \qquad a^{2312} = 0 \quad . \tag{21}$$

Equations (17.51), (17.52) and (17.46) yield

$$a^{2311} = 0 \quad , \qquad a^{2322} = 0 \quad , \qquad a^{1312} = 0 \quad . \tag{22}$$

Equation (17.36) reads

$$a^{3322} = a^{2211} \quad . \tag{23}$$

Equation (17.63) reads

$$a^{2233} = a^{1133} \quad . \tag{24}$$

Two equations (17.45)≡(17.54) read

$$a^{2323} = a^{1313} \quad . \tag{25}$$

Two equations (17.44)≡(17.55) read

$$a^{2313} = -a^{1323} \quad . \tag{26}$$

Equations (17.11), (17.12), (17.21) and (17.22) are linearly dependent. They yield three relations

$$a^{1211} = -a^{1112} \quad , \qquad a^{1222} = a^{1112} \quad , \qquad a^{2212} = -a^{1112} \quad . \tag{27}$$

Equation (17.66) is then satisfied and contains no additional information.

Equations (17.16), (17.26), (17.61) and (17.62) are linearly dependent. They yield three relations

$$a^{2222} = a^{1111} \quad , \qquad a^{2211} = a^{1122} \quad , \tag{28}$$

and

$$\boxed{a^{1212} = \tfrac{1}{2}\left(a^{1111} - a^{1122}\right) \quad .} \tag{29}$$

We insert relations (18)–(28) into definition (12) and obtain rotationally invariant stiffness matrix

$$a^{ijkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} a^{1111} & a^{1122} & a^{1133} & 0 & 0 & \underline{a}^{1112} \\ a^{1122} & a^{1111} & a^{1133} & 0 & 0 & -\underline{a}^{1112} \\ \underline{a}^{3311} & \underline{a}^{3311} & a^{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{1313} & -\underline{a}^{1323} & 0 \\ 0 & 0 & 0 & \underline{a}^{1323} & a^{1313} & 0 \\ -\underline{a}^{1112} & \underline{a}^{1112} & 0 & 0 & 0 & a^{1212} \end{pmatrix} \end{array} \quad , \tag{30}$$

where element a^{1212} is given by expression (29). The underlined elements are related to a possible non–symmetry of the stiffness matrix. The elements which are not underlined are well known from the rotationally invariant viscoelastic medium with a symmetric stiffness matrix. The underlines have no other meaning, i.e. $\underline{a}^{3311}$ is identical to a^{3311}.

The frequency–domain stiffness matrix of a viscoelastic medium is symmetric in the low–frequency and high–frequency limits (*Gurtin and Herrera, 1965, theorems 3.1 and 3.2*; *Christensen, 1971, Eqs 3.39–3.40*; *Fabrizio and Morro, 1988, corollaries 1 and 2; 1992, Eqs 3.28–3.29*; *Carcione, 2015, Eq. 2.24*). In the low–frequency and high–frequency limits, stiffness matrix (30) thus reduces to stiffness matrix

$$a^{ijkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \!\!\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} a^{1111} & a^{1122} & a^{1133} & 0 & 0 & 0 \\ a^{1122} & a^{1111} & a^{1133} & 0 & 0 & 0 \\ a^{1133} & a^{1133} & a^{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{1212} \end{pmatrix} \end{array} \tag{31}$$

specified by five complex–valued parameters. Stiffness matrix (31) is considerably different in the low–frequency and high–frequency limits.

5. GENERAL FORM OF A ROTATIONALLY INVARIANT NON–SYMMETRIC STIFFNESS MATRIX

We transform rotationally invariant non–symmetric stiffness matrix (30) from the coordinate system attached to the symmetry axis to a general coordinate system, and arrive at

$$\begin{aligned} a^{ijkl}(t_n) = {} & a^{1122}\,\delta_{ij}\delta_{kl} + a^{1212}\,(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + (a^{3333}-a^{1111})\,t_it_jt_kt_l \\ & + (a^{1133}-a^{1122})\,(\delta_{ij}-t_it_j)\,t_kt_l + (a^{3311}-a^{1122})\,t_it_j(\delta_{kl}-t_kt_l) \\ & + (a^{1313}-a^{1212})\,(\delta_{ik}t_jt_l+\delta_{il}t_jt_k+\delta_{jk}t_it_l+\delta_{jl}t_it_k-4\,t_it_jt_kt_l) \\ & + \tfrac{1}{2}\,a^{1112}\,(\delta_{ik}\varepsilon_{jlr}+\delta_{il}\varepsilon_{jkr}+\delta_{jk}\varepsilon_{ilr}+\delta_{jl}\varepsilon_{ikr})\,t_r \\ & + (a^{1323}-\tfrac{1}{2}\,a^{1112})\,(t_it_k\varepsilon_{jlr}+t_it_l\varepsilon_{jkr}+t_jt_k\varepsilon_{ilr}+t_jt_l\varepsilon_{ikr})\,t_r \quad , \end{aligned} \tag{32}$$

where a^{ijkl} on the right–hand side are the elements of stiffness matrix (30) expressed in the coordinate system in which the third coordinate axis coincides with the symmetry axis. Parameter a^{1212} on the right–hand side is given by expression (29).

Note that superscript 3 on the right–hand side of stiffness matrix (32) corresponds to symmetry axis t_n rather than to the third coordinate axis, analogously to stiffness matrix (30).

6. ISOTROPIC MEDIUM

An isotropic medium is invariant with respect to rotations about all three coordinate axes. We insert symmetry vector $t_n = (0,1,0)$ into stiffness tensor (32) and obtain stiffness matrix

$$a^{ijkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} \tilde{a}^{1111} & \tilde{a}^{1133} & \tilde{a}^{1122} & 0 & -\tilde{a}^{1112} & 0 \\ \tilde{a}^{3311} & \tilde{a}^{3333} & \tilde{a}^{3311} & 0 & 0 & 0 \\ \tilde{a}^{1122} & \tilde{a}^{1133} & \tilde{a}^{1111} & 0 & \tilde{a}^{1112} & 0 \\ 0 & 0 & 0 & \tilde{a}^{1313} & 0 & \tilde{a}^{1323} \\ \tilde{a}^{1112} & 0 & -\tilde{a}^{1112} & 0 & \tilde{a}^{1212} & 0 \\ 0 & 0 & 0 & -\tilde{a}^{1323} & 0 & \tilde{a}^{1313} \end{pmatrix} \end{array} \tag{33}$$

analogous to matrix (30) but corresponding to symmetry vector $t_a = (0,1,0)$. Note that superscript 3 on the right–hand side of stiffness matrix (33) corresponds to symmetry axis $t_a = (0,1,0)$ rather than the third coordinate axis, analogously to stiffness matrix (30). Comparing stiffness matrices (30) and (33), we obtain relations

$$a^{1111} = a^{3333} \quad , \qquad a^{1122} = a^{1133} \quad , \qquad a^{1122} = a^{3311} \tag{34}$$

and

$$a^{1212} = a^{1313} \quad , \qquad a^{1112} = 0 \quad , \qquad a^{1323} = 0 \quad . \tag{35}$$

Inserting identities (34) and (35) into stiffness matrix (30), we immediately see that the non–symmetric stiffness matrix of an isotropic medium reads

$$a^{ijkl} = \begin{matrix} & 11 & 22 & 33 & 23 & 13 & 12 \\ 11 & \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ 22 & \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ 33 & \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 23 & 0 & 0 & 0 & \mu & 0 & 0 \\ 13 & 0 & 0 & 0 & 0 & \mu & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 & \mu \end{matrix} , \tag{36}$$

and is thus identical to the standard symmetric stiffness matrix of an isotropic viscoelastic medium specified by two complex–valued parameters $\lambda = a^{1122} = c^{1122}/\varrho$ and $\mu = a^{1212} = c^{1212}/\varrho$.

We thus see that an isotropic viscoelastic medium invariant with respect to rotations about all three coordinate axes has always a symmetric stiffness matrix and cannot exhibit an "acoustical activity" analogous to the optical activity.

7. CONCLUSIONS

In the coordinate system where the third coordinate axis coincides with the symmetry axis, the non–symmetric stiffness matrix of a rotationally invariant viscoelastic medium has form (30). It is described by three additional complex–valued parameters in comparison with the symmetric stiffness matrix of a transversely isotropic (uniaxial) viscoelastic medium. During physical measurements, only these three additional complex–valued parameters can give rise to a possible asymmetry of the stiffness matrix of a rotationally invariant viscoelastic medium.

In a general coordinate system, the non–symmetric stiffness matrix of a rotationally invariant viscoelastic medium has form (32).

A rotationally invariant medium has frequently been studied in wave propagation under the assumption of the symmetric stiffness matrix. Our study reveals the differences due to abandoning this assumption. The frequency–domain ray theory for a non–symmetric stiffness matrix has been proposed by *Klimeš (2018b)*. The representation theorem for a non–symmetric stiffness matrix has been proposed by *Klimeš (2021)*.

Whereas the constitutive matrix of a biisotropic electromagnetic medium may be non–symmetric and may exhibit an optical activity (*Klimeš, 2017*), the stiffness matrix of an isotropic viscoelastic medium is symmetric and cannot exhibit an analogous "acoustical activity" which has been demonstrated in Section 6. This particular conclusion applies, e.g., to crude oil.

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