Two S-wave eigenvectors of the Christoffel matrix need not exist in anisotropic viscoelastic media

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ABSTRACT

The $3{\times}3{\times}3{\times}3$ *frequency–domain stiffness tensor is complex–valued in viscoelastic media. The* 3×3 *Christoffel matrix is then also complex–valued. Using a simple example, we demonstrate that a complex–valued Christoffel matrix need not have all three eigenvectors at an S–wave singularity, and we thus cannot apply the eigenvectors to calculating the phase–space derivatives of the Hamiltonian function.*

Keywords: attenuation, anisotropy, wave propagation, ray theory, ray tracing

1. INTRODUCTION

Attenuation is a very important phenomenon in wave propagation. The $3 \times 3 \times 3 \times 3$ frequency–domain stiffness tensor (elastic tensor, tensor of elastic moduli) is complex–valued in viscoelastic media. The 3×3 Christoffel matrix is then also complex–valued. The Hamiltonian function and its phase–space derivatives are usually calculated in terms of the eigenvectors of the Christoffel matrix (*Klimeš, 2006*). In this formulation, we need the S–wave eigenvectors of the Christoffel matrix even for calculating the geodesic deviation of P–wave rays.

In this paper, we demonstrate that a complex–valued Christoffel matrix need not have all three eigenvectors at an S–wave singularity, and we thus cannot apply the eigenvectors to calculating the phase–space derivatives of the Hamiltonian function. We present a simple example of a weakly anisotropic viscoelastic medium with three singular directions in Section 2, and calculate the corresponding Christoffel matrix in Section 3. We then show that the Christoffel matrix has just one S–wave eigenvector in each singular direction. Moreover, two S–wave eigenvectors of the Christoffel matrix need not exist even for arbitrarily weak attenuation in weakly anisotropic viscoelastic media. This simple but educational example resembles the case with the anisotropic ray theory not converging to the isotropic ray theory for vanishing anisotropy (*Bulant and Klimeš, 2002*).

Linear after–effect equations for a viscoelastic medium with fading memory in the time domain were proposed by *Boltzmann (1874)*. The linear relation between strain tensor $e_{ij} = e_{ij}(\mathbf{x}, t)$ of small deformations and stress tensor $\sigma_{ij} = \sigma_{ij}(\mathbf{x}, t)$ at point $\mathbf{x}$ and time t in anisotropic viscoelastic medium reads (*Volterra, 1909, Eq. III*)

$$\sigma_{ij}(\mathbf{x}, t) = \sum_{k,l} R^0_{ijkl}(\mathbf{x})\, e_{kl}(\mathbf{x}, t) + \int_{-\infty}^{t} \mathrm{d}\xi \sum_{k,l} \tfrac{\partial}{\partial t} R_{ijkl}(\mathbf{x}, t-\xi)\, e_{kl}(\mathbf{x}, \xi) \quad , \tag{1}$$

where $R_{ijkl} = R_{ijkl}(\mathbf{x}, \Delta t)$ represents the $3\times3\times3\times3$ real–valued relaxation tensor at point $\mathbf{x}$ after time Δt. The Fourier transform of this linear constitutive equation according to the convention of *Červený (2001, Eq. A.1.2)*, reads

$$\sigma_{ij}(\mathbf{x}, \omega) = \sum_{k,l} c_{ijkl}(\mathbf{x}, \omega)\, e_{kl}(\mathbf{x}, \omega) \quad , \tag{2}$$

where the complex–valued frequency–domain viscoelastic stiffness tensor reads

$$c_{ijkl}(\mathbf{x}, \omega) = R^0_{ijkl}(\mathbf{x}) + \int_0^{+\infty} \mathrm{d}t\ \exp(\mathrm{i}\omega t)\, \tfrac{\partial}{\partial t} R_{ijkl}(\mathbf{x}, t) \quad . \tag{3}$$

This paper represents a gentle mention that the equations designed for a real–valued stiffness tensor need not be applicable to a complex–valued stiffness tensor.

2. EXAMPLE OF A STIFFNESS TENSOR

Let us start with simple example

$$a_0^{ijkl} = \begin{matrix} & 11 & 22 & 33 & 23 & 13 & 12 \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{matrix} \begin{pmatrix} 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & & 3-4\mathrm{i}\delta & 0 & 0 & 0 \\ & & & 1-3\mathrm{i}\delta & 0 & 0 \\ & & & & 1-3\mathrm{i}\delta & 0 \\ & & & & & 1-3\mathrm{i}\delta \end{pmatrix} \tag{4}$$

of the complex–valued frequency–domain density–normalized stiffness tensor of an isotropic viscoelastic medium. Since the stiffness tensor is symmetric with respect to the first pair of indices and with respect to the second pair of indices, it is expressed in the form of the 6×6 stiffness matrix whose lines correspond to the first pair of indices and columns to the second pair of indices. In this paper, we assume that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices, i.e., that 6×6 stiffness matrix (4) is symmetric. For the sake of simplicity, we have omitted here a multiplication factor which roughly corresponds to the square of S–wave velocity. Parameter δ is assumed real–valued. Considering here the Fourier transform with the sign convention according to *Červený (2001, Eq. A.1.2)*, parameter δ is taken positive. The S–wave quality factor then reads $Q_\mathrm{S} = 1/(3\delta)$.

This is a frequent type of an isotropic viscoelastic medium, with the square of P–wave velocity roughly three times greater than the square of S–wave velocity. The medium is elastic in volume deformations (*Anderson, Ben Menahem and Archambeau, 1965*). The medium is weakly attenuating for small δ.

We now modify stiffness tensor (4) by a small anisotropic perturbation,

$$a^{ijkl} = \begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & & 3-4\mathrm{i}\delta & 0 & 0 & 0 \\ & & & 1-3\mathrm{i}\delta+2\epsilon & \mathrm{i}\epsilon & 2\mathrm{i}\epsilon \\ & & & & 1-3\mathrm{i}\delta & \mathrm{i}\epsilon \\ & & & & & 1-3\mathrm{i}\delta-2\epsilon \end{pmatrix} \end{array} , \qquad (5)$$

where ϵ is a small complex–valued parameter.

The material stability conditions (*Fabrizio and Morro, 1992, Eqs 4.3.8–4.3.9*) require the real part of frequency–domain stiffness matrix (3) to be positive–semidefinite, and the imaginary part to be negative–semidefinite. In our example of stiffness matrix (5), the stability conditions imply non–negative δ,

$$\delta \geq 0 \quad , \qquad (6)$$

and inequalities

$$4\,[\mathrm{Re}\,(\epsilon)]^2 + 6\,[\mathrm{Im}\,(\epsilon)]^2 \leq 3 \quad , \qquad 4\,[\mathrm{Re}\,(\epsilon)]^2 + 6\,[\mathrm{Im}\,(\epsilon)]^2 + 4\,[\mathrm{Im}\,(\epsilon)]^3 \leq 1 \quad , \qquad (7)$$

$$4\left[\frac{\mathrm{Im}\,(\epsilon)}{3\,\delta}\right]^2 + 6\left[\frac{\mathrm{Re}\,(\epsilon)}{3\,\delta}\right]^2 \leq 3 \quad , \qquad 4\left[\frac{\mathrm{Im}\,(\epsilon)}{3\,\delta}\right]^2 + 6\left[\frac{\mathrm{Re}\,(\epsilon)}{3\,\delta}\right]^2 + 4\left[\frac{\mathrm{Re}\,(\epsilon)}{3\,\delta}\right]^3 \leq 1 \quad . \qquad (8)$$

Positive real–valued parameter δ is small for usual attenuation. Then $\mathrm{Re}\,(\epsilon)$ and $\mathrm{Im}\,(\epsilon)$ should also be small.

3. CHRISTOFFEL MATRIX

The 3×3 Christoffel matrix is defined as

$$\Gamma^{ik} = a^{ijkl} p_j p_l \quad , \qquad (9)$$

where p_j are the components of the slowness vector. For the real–valued reference slowness vector, we obtain the complex–valued reference Christoffel matrix.

The Christoffel matrix in isotropic viscoelastic medium (4) reads

$$\Gamma_0^{ik} = \left[(1-3\mathrm{i}\delta)\,\delta^{ik} p_l p_l + (2-\mathrm{i}\delta) p_i p_k\right] \quad . \qquad (10)$$

The Christoffel matrix in medium (5) reads

$$\Gamma^{ik} = \Gamma_0^{ik} + \epsilon \begin{pmatrix} \mathrm{i}2p_2p_3-2p_2p_2 & \mathrm{i}(p_3p_3+2p_2p_3+p_1p_3)-2p_1p_2 & \mathrm{i}(p_2p_3+2p_2p_2+p_1p_2) \\ & \mathrm{i}4p_1p_3+2p_3p_3-2p_1p_1 & \mathrm{i}(p_1p_3+2p_1p_2+p_1p_1)+2p_2p_3 \\ & & \mathrm{i}2p_1p_2+2p_2p_2 \end{pmatrix} . \qquad (11)$$

Christoffel matrix (11) indicates at least three singular directions of medium (5). These three singular directions correspond to the coordinate axes.

3.1. First singular direction

For $p_j = (p, 0, 0)$, Christoffel matrix (11) reads

$$\Gamma^{ik} = \begin{pmatrix} 3-4\mathrm{i}\delta & 0 & 0 \\ & 1-3\mathrm{i}\delta-2\epsilon & \mathrm{i}\epsilon \\ & & 1-3\mathrm{i}\delta \end{pmatrix} p^2 \quad . \tag{12}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(1, 0, 0)$, and double S–wave eigenvalue $(1-3\mathrm{i}\delta-\epsilon)p^2$ with just a single eigenvector $(0, 1, -\mathrm{i})$ corresponding to right–handed circular polarization.

The two–fold S–wave slowness surface in a vicinity of the S–wave singularity looks like the complex–valued square root above the complex plane.

The S–wave eigenvectors in a vicinity of the S–wave singularity correspond to two right–handed elliptical polarizations close to the circular polarization. One eigenvector smoothly changes to the other along a closed path around the S–wave singularity.

3.2. Second singular direction

For $p_j = (0, p, 0)$, Christoffel matrix (11) reads

$$\Gamma^{ik} = \begin{pmatrix} 1-3\mathrm{i}\delta-2\epsilon & 0 & 2\mathrm{i}\epsilon \\ & 3-4\mathrm{i}\delta & 0 \\ & & 1-3\mathrm{i}\delta+2\epsilon \end{pmatrix} p^2 \quad . \tag{13}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(0, 1, 0)$, and double S–wave eigenvalue $(1-3\mathrm{i}\delta)p^2$ with just a single eigenvector $(1, 0, -\mathrm{i})$ corresponding to left–handed circular polarization.

The two–fold S–wave slowness surface in a vicinity of the S–wave singularity looks like the complex–valued square root above the mirror imaged complex plane.

The S–wave eigenvectors in a vicinity of the S–wave singularity correspond to two left–handed elliptical polarizations close to the circular polarization. One eigenvector smoothly changes to the other along a closed path around the S–wave singularity.

3.3. Third singular direction

For $p_j = (0, 0, p)$, Christoffel matrix (11) reads

$$\Gamma^{ik} = \begin{pmatrix} 1-3\mathrm{i}\delta & \mathrm{i}\epsilon & 0 \\ & 1-3\mathrm{i}\delta+2\epsilon & 0 \\ & & 3-4\mathrm{i}\delta \end{pmatrix} p^2 \quad . \tag{14}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(0, 0, 1)$, and double S–wave eigenvalue $(1-3\mathrm{i}\delta+\epsilon)p^2$ with just a single eigenvector $(1, -\mathrm{i}, 0)$ corresponding to right–handed circular polarization.

The two–fold S–wave slowness surface in a vicinity of the S–wave singularity looks like the complex–valued square root above the complex plane.

The S–wave eigenvectors in a vicinity of the S–wave singularity correspond to two right–handed elliptical polarizations close to the circular polarization. One eigenvector smoothly changes to the other along a closed path around the S–wave singularity.

4. CONCLUSIONS

The ray tracing equations and the equations of geodesic deviation in heterogeneous anisotropic media are often formulated using the eigenvectors of the Christoffel matrix (*Klimeš, 2006*). The presented example demonstrates that two S–wave eigenvectors of the Christoffel matrix need not exist in weakly anisotropic viscoelastic media, even for arbitrarily weak attenuation.

It is thus necessary to formulate the ray tracing equations and the corresponding equations of geodesic deviation using the eigenvalues of the Christoffel matrix, without the eigenvectors of the Christoffel matrix (*Klimeš, 2020*).

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