JOURNAL OF ELECTROMAGNETIC WAVES AND APPLICATIONS, 2017
https://doi.org/10.1080/09205071.2017.1356244



# Superposition of Gaussian packets in smoothly heterogeneous bianisotropic media

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**ABSTRACT**

We derive the equations for the integral superposition of Gaussian packets in smoothly heterogeneous media. The superposition of Gaussian packets is especially useful in the vicinity of spatial caustics where the ray theory under consideration is singular. The equations are applicable to both the anisotropic ray theory and the coupling ray theory in bianisotropic media, or to the isotropic ray theory in isotropic media. The equations can be used in both Cartesian and curvilinear coordinates. The equations are applicable not only to electromagnetic waves in bianisotropic media, but also to other kinds of vectorial waves, e.g. to elastic waves in generally anisotropic media.

**ARTICLE HISTORY**
Received 6 February 2017
Accepted 7 May 2017

**KEYWORDS**
Summation of Gaussian packets; bianisotropy; heterogeneous media; wave propagation; travel time; amplitude; Green tensor; wave coupling

## 1. Introduction

The equations for the integral superposition of Gaussian packets in smoothly heterogeneous isotropic media were derived using the Maslov asymptotic theory applied to a general 3-D subspace of the 6-D complex phase space by Klimeš [1], and demonstrated in the computation of elastic wave fields by Klimeš [2]. In this paper, we rederive the equations for the integral superposition of Gaussian packets in smoothly heterogeneous *bianisotropic media*. The presented equations are applicable not only to electromagnetic waves in bianisotropic media, but also to other kinds of vectorial waves, e.g. to elastic waves in generally anisotropic media.

The presented integral superposition of Gaussian packets may correspond to the anisotropic ray theory [3], to the frequency-dependent coupling ray theory [4], or to the prevailing–frequency approximation of the coupling ray theory [5,6] in bianisotropic media, or to the isotropic ray theory in isotropic media.

The lower-case indices $i, j, \dots$ take the values of 1, 2, 3. The Einstein summation convention over repeated indices is used.

## 2. Gaussian packets

We consider Cartesian or curvilinear coordinates $x^i$ in 3-D space. We consider an orthonomic system $x^i = \tilde{x}^i(\gamma^a)$ of rays parametrized by ray coordinates $\gamma^a$, where $\gamma^1$ and $\gamma^2$ are the ray parameters, and $\gamma^3$ is the parameter along rays determined by the form of the

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Hamiltonian function. In the case of the coupling ray theory, the orthonomic system of rays is represented by the system of reference rays along which the coupling equations are solved. The reference rays are calculated using the reference Hamiltonian function.

The Gaussian packet centred at point $\tilde{x}^i$ reads

$$\mathbf{g}(x^m,\omega) = \mathbf{A}^{\mathrm{GP}}(\tilde{x}^m,\omega)\,\exp\left\{\mathrm{i}\omega\left[\tau(\tilde{x}^n) + (x^k-\tilde{x}^k)\,p_k(\tilde{x}^n) + \tfrac{1}{2}(x^k-\tilde{x}^k)\,f_{kl}(\tilde{x}^n)\,(x^l-\tilde{x}^l)\right]\right\}, \quad (1)$$

where $\omega$ is the circular frequency, $\tau(\tilde{x}^n)$ is the travel time corresponding to the orthonomic system of rays at point $\tilde{x}^i$, $p_k(\tilde{x}^n)$ is the slowness vector corresponding to the orthonomic system of rays at point $\tilde{x}^i$, $f_{kl}(\tilde{x}^n)$ is the complex-valued matrix with positive imaginary part describing the shape of the Gaussian packet centred at point $\tilde{x}^i$, and $\mathbf{A}^{\mathrm{GP}}(\tilde{x}^n,\omega)$ is the complex-valued vectorial or tensorial amplitude of the Gaussian packet centred at point $\tilde{x}^i$. The amplitudes of Gaussian packets corresponding to a particular source are usually vectorial. The amplitudes of Gaussian packets designed to compose the Green tensor are usually tensorial.

For the electromagnetic ray theory developed in terms of the magnetic vector potential with the Weyl gauge (zero electric potential) according to Klimeš [3], the amplitudes are covariant spatial 3-vectors or $3\times3$ tensors. Without the Weyl gauge, these amplitudes would be covariant space-time 4-vectors or $4\times4$ tensors. For the elastic ray theory, the amplitudes would be spatial 3-vectors or $3\times3$ tensors, etc.

The vectorial or tensorial amplitudes may correspond to the anisotropic ray theory [3], to the frequency-dependent coupling ray theory [4], or to the prevailing–frequency approximation of the coupling ray theory [5,6] in bianisotropic media, or to the isotropic ray theory in isotropic media.

Matrix $f_{kl}(\tilde{x}^n)$ may be chosen arbitrarily, but must smoothly vary with coordinates $\tilde{x}^n$. Matrix $f_{kl}(\tilde{x}^n)$ need not satisfy the equations for Gaussian packets propagating along the rays, because Gaussian packets arriving at different points of the same ray may correspond to different initial conditions for the shape of the Gaussian packets. Matrix $f_{kl}(\tilde{x}^n)$ may also depend on frequency $\omega$. In the superposition of Gaussian packets, vectorial or tensorial amplitude $\mathbf{A}^{\mathrm{GP}}(\tilde{x}^n,\omega)$ will be determined by the choice of matrix $f_{kl}(\tilde{x}^n)$.

## 3. Integral superposition of Gaussian packets

The time-harmonic superposition of Gaussian packets at frequency $\omega$ reads

$$\mathbf{u}(x^m,\omega) = \iiint \mathrm{d}\tilde{x}^1\mathrm{d}\tilde{x}^2\mathrm{d}\tilde{x}^3\,\mathbf{A}^{\mathrm{GP}}(\tilde{x}^m,\omega) \times \exp\left\{\mathrm{i}\omega\left[\tau(\tilde{x}^m) + (x^k-\tilde{x}^k)\,p_k(\tilde{x}^m) + \tfrac{1}{2}(x^k-\tilde{x}^k)\,f_{kl}(\tilde{x}^m)\,(x^l-\tilde{x}^l)\right]\right\}. \quad (2)$$

For very high frequencies $\omega$, the Gaussian packets are very narrow and we can calculate superposition integral (2) analytically.

The paraxial ray approximation of travel time $\tau(\tilde{x}^n)$ centred at point $x^i$ reads

$$\tau(\tilde{x}^n) \approx \tau(x^n) + (\tilde{x}^k-x^k)\,p_k(x^m) + \tfrac{1}{2}(\tilde{x}^k-x^k)\,N_{kl}(x^m)\,(\tilde{x}^l-x^l), \quad (3)$$

where $N_{kl}(x^m)$ is the matrix of the second-order partial derivatives of travel time at point $x^i$. The paraxial ray approximation of slowness vector $p_k(\tilde{x}^n)$ centred at point $x^i$ reads

$$p_i(\tilde{x}^n) \approx p_i(x^n) + N_{ik}(x^m)\,(\tilde{x}^k - x^k). \tag{4}$$

For very high frequencies $\omega$, the Gaussian packets are very narrow and we may apply approximation

$$f_{kl}(\tilde{x}^n) \approx f_{kl}(x^n) \tag{5}$$

of smoothly varying matrix $f_{kl}(\tilde{x}^n)$, and approximation

$$\mathbf{A}^{\mathrm{GP}}(\tilde{x}^m, \omega) \approx \mathbf{A}^{\mathrm{GP}}(x^m, \omega) \tag{6}$$

of amplitude $\mathbf{A}^{\mathrm{GP}}(\tilde{x}^m, \omega)$.

We insert paraxial ray approximations (3)–(6) into integral superposition (2) and obtain

$$\mathbf{u}(x^m, \omega) \approx \iiint \mathrm{d}\tilde{x}^1 \mathrm{d}\tilde{x}^2 \mathrm{d}\tilde{x}^3\, \mathbf{A}^{\mathrm{GP}}(\tilde{x}^m, \omega) \times \exp\left\{\mathrm{i}\omega\left[\tau(x^m) + \tfrac{1}{2}(\tilde{x}^k - x^k)\,[f_{kl}(x^m) - N_{ik}(x^m)]\,(\tilde{x}^l - x^l)\right]\right\}. \tag{7}$$

We now calculate integral (7) analytically, and obtain

$$\mathbf{u}(x^m, \omega) \approx \mathbf{A}^{\mathrm{GP}}(x^m, \omega)\left(\frac{2\pi}{\omega}\right)^{\frac{3}{2}} \left\{\sqrt{\det}\left\{\mathrm{i}\left[N_{ab}(x^n) - f_{ab}(x^n)\right]\right\}\right\}^{-1} \exp\left[\mathrm{i}\omega\tau(x^m)\right], \tag{8}$$

where function $\sqrt{\det}\left(M_{ab}\right)$ is the product of the square roots of the eigenvalues of matrix $M_{ab}$. The individual square roots are taken with positive real parts.

For high frequencies $\omega$, we need

$$\mathbf{u}(x^m, \omega) \approx \mathbf{A}(x^m, \omega)\; \exp\left[\mathrm{i}\omega\tau(x^m)\right], \tag{9}$$

where $\mathbf{A}(x^m, \omega)$ is the complex-valued vectorial or tensorial *ray–theory amplitude*.

In bianisotropic media, vectorial or tensorial ray–theory amplitude $\mathbf{A}(x^m, \omega)$ may represent either the anisotropic–ray–theory vectorial or tensorial amplitude, or the frequency-dependent coupling–ray–theory vectorial or tensorial amplitude calculated using the scalar anisotropic–ray–theory amplitude corresponding to the orthonomic system of rays, or the vectorial or tensorial amplitude of the prevailing–frequency approximation of the coupling ray theory [5,6]. In isotropic media, vectorial or tensorial amplitude $\mathbf{A}(x^m, \omega)$ represents the isotropic–ray–theory vectorial or tensorial amplitude.

Comparing expressions (8) and (9), we obtain

$$\mathbf{A}^{\mathrm{GP}}(x^m, \omega) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}} \mathbf{A}(x^m, \omega)\,\sqrt{\det}\left\{\mathrm{i}\left[N_{ab}(x^n) - f_{ab}(x^n)\right]\right\}. \tag{10}$$

Inserting (10) into (2), we obtain the time-harmonic superposition

$$\mathbf{u}(x^m,\omega) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}} \iiint \mathrm{d}\tilde{x}^1\mathrm{d}\tilde{x}^2\mathrm{d}\tilde{x}^3\, \mathbf{A}(\tilde{x}^m,\omega)\, \sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^n)-f_{ab}(\tilde{x}^n)]\}}$$
$$\times \exp\{\mathrm{i}\omega[\tau(\tilde{x}^n) + (x^k-\tilde{x}^k)\,p_k(\tilde{x}^n) + \tfrac{1}{2}(x^k-\tilde{x}^k)\,f_{kl}(\tilde{x}^n)\,(x^l-\tilde{x}^l)]\} \tag{11}$$

of Gaussian packets.

Since the integral superposition of Gaussian packets is usually calculated numerically in ray coordinates, we shall transfer superposition (11) to ray coordinates. We define transformation matrix

$$X^k_a = \frac{\partial x^k}{\partial \gamma^a}. \tag{12}$$

The paraxial matrix $X^k_a$ of geometrical spreading can be calculated using the Hamiltonian equations of geodesic deviation (dynamic ray tracing equations, paraxial ray tracing equations) derived by Červený [7].

We change the integration variables in integral superposition (11) to ray coordinates and arrive at

$$\mathbf{u}(x^m,\omega) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}} \iiint \mathrm{d}\gamma^1\mathrm{d}\gamma^2\mathrm{d}\gamma^3\, \mathbf{A}(\tilde{x}^m,\omega)\, \left|\det[X^g_a(\tilde{x}^r)]\right| \sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^h)-f_{ab}(\tilde{x}^h)]\}}$$
$$\times \exp\{\mathrm{i}\omega[\tau(\tilde{x}^n) + (x^k-\tilde{x}^k)\,p_k(\tilde{x}^n) + \tfrac{1}{2}(x^k-\tilde{x}^k)\,f_{kl}(\tilde{x}^n)\,(x^l-\tilde{x}^l)]\}\,, \tag{13}$$

where $\tilde{x}^n = \tilde{x}^n(\gamma^a)$. The vectorial or tensorial ray–theory amplitude **A** corresponds to the ray theory under consideration, e.g. to the anisotropic ray theory [3], to the coupling ray theory [4], to the prevailing–frequency approximation of the coupling ray theory [5,6], or to the isotropic ray theory. Integral superposition (13) of Gaussian packets is applicable to electromagnetic waves in smoothly heterogeneous bianisotropic media, but also to other kinds of vectorial waves, e.g. to elastic waves in generally anisotropic media.

Whereas factors $\mathbf{A}(\tilde{x}^m,\omega)$ and $\sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^n)-f_{ab}(\tilde{x}^n)]\}}$ in (13) are not smooth in the vicinity of caustics, product

$$\mathbf{A}(\tilde{x}^m,\omega)\,\left|\det[X^g_a(\tilde{x}^r)]\right| \sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^h)-f_{ab}(\tilde{x}^h)]\}} \tag{14}$$

is smooth everywhere in its definition domain.

## 4. Note on the ray–theory amplitude

We may decompose vectorial or tensorial amplitude **A** into complex-valued scalar amplitude $A$ and complex-valued polarization vector or tensor **e**,

$$\mathbf{A}(\tilde{x}^m,\omega) = A(\tilde{x}^m,\omega)\,\mathbf{e}(\tilde{x}^m,\omega). \tag{15}$$

The complex-valued scalar amplitude corresponds to the orthonomic system of rays, and may be expressed as [3, Equation 95]

$$A(\tilde{x}^m,\omega) = C(\tilde{x}^m,\omega)\sqrt{\frac{1}{\varrho(\tilde{x}^h)}\frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3}(\tilde{x}^h)\,\frac{1}{|\det(X^i_a)|}}\;\exp[\mathrm{i}\varphi(\tilde{x}^h)], \tag{16}$$

where $\varrho(\tilde{x}^h)$ is defined by Klimeš [3, Equation 84]. Here, $\varphi$ is the phase shift due to caustics, due to the gradient of optical activity [3, Equations 95, 115], etc. Complex-valued reduced scalar amplitude $C$ is constant along the ray in smooth parts of a medium, and is determined by the initial conditions. Reduced amplitude $C$ obviously depends on the selection of ray parameters $\gamma^1$ and $\gamma^2$.

Complex-valued polarization vector or tensor $\mathbf{e}$ depends on the eigenvectors of the Kelvin–Christoffel matrix [3]. It also depends on the coupling equations, if we consider the coupling ray theory or the isotropic ray theory.

We insert decomposition (15) with scalar amplitude (16) into integral superposition (13) and obtain

$$\begin{aligned}\mathbf{u}(x^m,\omega) = \left(\frac{\omega}{2\pi}\right)^{\frac{3}{2}}\iiint \mathrm{d}\gamma^1\mathrm{d}\gamma^2\mathrm{d}\gamma^3\;\mathbf{e}(\tilde{x}^m,\omega)\;C(\tilde{x}^m)\;\sqrt{\frac{1}{\varrho(\tilde{x}^h)}\frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3}(\tilde{x}^h)}\\ \times\sqrt{|\det[X^c_d(\tilde{x}^r)]|}\;\sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^h)-f_{ab}(\tilde{x}^h)]\}}\;\exp[\mathrm{i}\varphi(\tilde{x}^h)]\\ \times\exp\{\mathrm{i}\omega[\tau(\tilde{x}^n)+(x^k-\tilde{x}^k)\,p_k(\tilde{x}^n)+\tfrac{1}{2}(x^k-\tilde{x}^k)\,f_{kl}(\tilde{x}^n)\,(x^l-\tilde{x}^l)]\}\,.\end{aligned} \tag{17}$$

Whereas factors $\sqrt{\det}\{\mathrm{i}[N_{ab}(\tilde{x}^n)-f_{ab}(\tilde{x}^n)]\}$ and $\exp[\mathrm{i}\varphi(\tilde{x}^h)]$ in (17) are not smooth in the vicinity of caustics, product

$$\sqrt{|\det[X^c_d(\tilde{x}^r)]|}\;\sqrt{\det\{\mathrm{i}[N_{ab}(\tilde{x}^h)-f_{ab}(\tilde{x}^h)]\}}\;\exp[\mathrm{i}\varphi(\tilde{x}^h)] \tag{18}$$

is smooth everywhere in its definition domain.

## 5. Conclusions

Integral superposition (13) of Gaussian packets is applicable to electromagnetic waves in smoothly heterogeneous parts of bianisotropic media, but also to other kinds of vectorial waves, e.g. to elastic waves in generally anisotropic media. The smoothly heterogeneous parts of bianisotropic media considered in superposition (13) may be separated from the source region by several smooth interfaces.

The superposition of Gaussian packets is especially useful in the vicinity of spatial caustics where the ray theory under consideration is singular. The equations are applicable to regularize both the anisotropic ray theory and the coupling ray theory in bianisotropic media, or the isotropic ray theory in isotropic media.

The integral superposition of Gaussian beams can be derived from the integral superposition of Gaussian packets by means of high-frequency asymptotic quadrature with respect to ray coordinate $\gamma^3$ along rays [8].

## Disclosure statement

No potential conflict of interest was reported by the author.

## Funding

The research has been supported by the Grant Agency of the Czech Republic under contracts 16-01312S and 16-05237S, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

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