A note on two-point paraxial travel times

Vlastislav Červený

Charles University in Prague, Faculty of Mathematics and Physics,
Department of Geophysics, Ke Karlovu 3, 121 16 Praha 2, Czech Republic
(vcerveny@seis.karlov.mff.cuni.cz)

Received: October 2, 2012; Revised: January 12, 2013; Accepted: February 15, 2013

ABSTRACT

Recently, several expressions for the two-point paraxial travel time in laterally varying, isotropic or anisotropic layered media were derived. The two-point paraxial travel time gives the travel time from point S' to point R', both these points being situated close to a known reference ray Ω, along which the ray-propagator matrix was calculated by dynamic ray tracing. The reference ray and the position of points S' and R' are specified in Cartesian coordinates. Two such expressions for the two-point paraxial travel time play an important role. The first is based on the 4×4 ray propagator matrix, computed by dynamic ray tracing along the reference ray in ray-centred coordinates. The second requires the knowledge of the 6×6 ray propagator matrix computed by dynamic ray tracing along the reference ray in Cartesian coordinates. Both expressions were derived fully independently, using different methods, and are expressed in quite different forms. In this paper we prove that the two expressions are fully equivalent and can be transformed into each other.

Keywords: seismic anisotropy, theoretical seismology, wave propagation, paraxial ray methods

1. INTRODUCTION

We consider a three-dimensional, laterally varying, isotropic or anisotropic layered medium and the eikonal equation in Hamiltonian form $\mathcal{H}(x_i, p_j) = 1/2$, where x_i are Cartesian coordinates of the position vector $\mathbf{x}$ and p_j the Cartesian components of slowness vector $\mathbf{p}$. We use Hamiltonians $\mathcal{H}(x_i, p_j)$ which are homogeneous functions of the second degree in p_j. Using the ray tracing equations $\mathrm{d}x_i/\mathrm{d}\tau = \partial\mathcal{H}/\partial p_i$, $\mathrm{d}p_i/\mathrm{d}\tau = -\partial\mathcal{H}/\partial x_i$ (with the initial conditions $x_i(\tau_0)$, $p_i(\tau_0)$), we compute a reference ray Ω, specified by $x_i(\tau)$ with slowness vector $p_i(\tau)$ at any point of the reference ray. The quantity τ is a monotonic parameter along the reference ray. For the Hamiltonian considered in this paper, which is a homogeneous function of the second degree in p_j, parameter τ represents the travel time along the reference ray. As a by-product of ray-tracing, we also obtain $u_i = \partial\mathcal{H}/\partial p_i$ and $\eta_i = -\partial\mathcal{H}/\partial x_i$, the Cartesian components of the ray-velocity vector $\mathbf{u}$ and of vector $\boldsymbol{\eta}$ at any point of the reference ray. Together with ray tracing of the reference ray, or after it, we can also perform dynamic ray tracing, and compute the ray propagator matrix. The dynamic ray tracing can be performed in various coordinate systems. In this paper, we consider two useful coordinate systems, namely the Cartesian coordinate system x_1, x_2, x_3, and the ray-centred coordinate system q_1, q_2, q_3. Whereas the Cartesian coordinate system is orthonormal, the ray-centred coordinate system q_1, q_2, q_3 is non-orthogonal. The basic property of the ray-centred coordinate system is that reference ray Ω represents the q_3-coordinate axis of the system. The remaining two coordinates q_1, q_2 are orthonormal in the plane tangent to the wavefront at the point, where the wavefront intersects reference ray Ω. The dynamic ray tracing system is linear. For this reason, we can construct the fundamental matrix of the dynamic ray tracing system, consisting of its linearly independent solutions. The ray-propagator matrix is the fundamental matrix of the dynamic ray tracing system, which equals the identity matrix at some selected point of the ray. In Cartesian coordinates, the ray-propagator matrix has $6 \times 6 = 36$ elements. In ray-centred coordinates, many of these elements vanish or are constant, so that their number reduces to $4 \times 4 = 16$. This is one of the advantages of dynamic ray tracing in ray-centred coordinates. For more details on ray tracing, dynamic ray tracing and ray propagator matrices see *Červený (2001)*.

We choose arbitrarily two points S and R on the reference ray, and two points S' and R', close to S and R, respectively. We specify the position of points S, R, S' and R' in Cartesian coordinates. The travel time from S to R, denoted by $T(R,S)$, is known from ray tracing. The problem of estimating the travel time from S' to R', called the two-point paraxial travel time and denoted $T(R',S')$, was solved by *Červený et al. (2012)*, using the 4×4 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ computed by dynamic ray tracing along the reference ray in ray-centred coordinates and using the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$, computed by dynamic ray tracing in Cartesian coordinates. The derivation of $T(R',S')$ using both approaches was quite different. In the method based on the 4×4 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$, the most important step consisted in the transformation of the second derivatives of the travel time from ray-centred to Cartesian coordinates (*Červený and Klimeš, 2010*). In the method based on the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$, the most important step consisted in the extension of the classical Hamilton's theory by the equations of geodesic deviations (*Hamilton, 1837; Klimeš, 2009*). Both derivations are independent; the resulting equations are quite different and use different ray-propagator matrices.

For this reason, it would be very interesting to try to derive the relation between these two expressions for $T(R',S')$ analytically. In this paper, we offer a method of performing this. The method is based on the direct transformation of 6×6 ray-propagator matrices in Cartesian and ray-centred coordinates. First we use the transformation between the 6×6 ray propagator matrix in Cartesian coordinates and ray-centred coordinates given by *Červený (2001, Eq. 4.3.38)*. We then reduce the 6×6 ray-propagator matrix in ray-centred coordinates to its 4×4 version. In this way, we can obtain $T(R', S')$, expressed in terms of the 6×6 ray-propagator matrix in Cartesian coordinates and $T(R', S')$, expressed in terms of the 4×4 ray propagator matrix in ray-centred coordinates. As a result, we prove that both expressions for $T(R', S')$ are identical. We can thus use alternatively any of them.

Briefly to the content of the paper. In Section 2, we present and explain the formula for the two-point paraxial travel time $T(R', S')$, based on the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R, S)$ computed by dynamic ray tracing in global Cartesian coordinates. In Section 3, we discuss the transformation of the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R, S)$ in Cartesian coordinates into the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R, S)$ in ray-centred coordinates and show how the 6×6 ray propagator matrix $\mathbf{\Pi}^{(q)}(R, S)$ can be reduced to the 4×4 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R, S)$. In Section 4, we insert the relation between the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R, S)$ and the 4×4 ray propagator matrix $\mathbf{\Pi}^{(q)}(R, S)$ into the expression for the two-point paraxial travel time $T(R', S')$ presented in Section 2. In this way, we obtain a new formula for the two-point paraxial travel time $T(R', S')$, expressed in terms of the 4×4 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R, S)$, computed by dynamic ray tracing in ray-centred coordinates. We show that this formula is exactly the same as the analogous formula derived by *Červený et al. (2012)* using a quite different method.

We use both component and matrix notations. In the component notation, the upper-case indices $(I, J, K, ...)$ take the values 1 and 2, and the lower case indices $(i, j, k, ...)$ the values 1,2,3. The Einstein summation convention is used. In the matrix notation, the square matrices are denoted by bold upright capital symbols and the 3×2 matrices by bold upright capital calligraphic symbols. We use notation $\mathbf{A}^{-1}$ for the inverse of matrix $\mathbf{A}$, and $\mathbf{A}^{-T}$ for the transpose of the inverse of $\mathbf{A}$. The bold upright capital symbols with hats denote 3×3 matrices. The superscripts (x) and (q) are used to denote the components of the matrices in Cartesian and ray-centred coordinates, respectively. Whenever there may be reason for confusion, the dimensions of the matrices are indicated. The vectors in matrix notation are denoted by bold upright lower-case symbols. They are represented by 3×1 column matrices, the components of which are given in Cartesian coordinates. The scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$ reads $\mathbf{a}^T\mathbf{b}$, and the dyadics $\mathbf{a}\mathbf{b}^T$.

2. TWO-POINT PARAXIAL TRAVEL TIMES $T(R', S')$ IN TERMS OF 6×6 RAY PROPAGATOR MATRIX $\mathbf{\Pi}^{(x)}(R, S)$

The formula for the two-point paraxial travel time $T(R', S')$ expressed in terms of the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(x)}(R, S)$, computed by dynamic ray tracing in global Cartesian coordinates x_i, was derived in *Červený et al. (2012, Eq. 47)*. It reads:

$$\begin{aligned} T(R', S') = {} & T(R, S) + \delta x_i^R p_i(R) - \delta x_i^S p_i(S) \\ & - \tfrac{1}{2} T^{-1}(R, S)[\delta x_i^R p_i(R) - \delta x_i^S p_i(S)]^2 + \tfrac{1}{2}\delta x_i^R (\widehat{\mathbf{P}}_2^{(x)} \widehat{\mathbf{Q}}_2^{(x)-1})_{ij} \delta x_j^R \\ & + \tfrac{1}{2}\delta x_i^S (\widehat{\mathbf{Q}}_2^{(x)-1} \widehat{\mathbf{Q}}_1^{(x)})_{ij} \delta x_j^S - \delta x_i^S (\widehat{\mathbf{Q}}_2^{(x)-1})_{ij} \delta x_j^R \quad . \end{aligned} \tag{1}$$

Here

$$\delta x_i^R = x_i(R') - x_i(R) \ , \qquad \delta x_i^S = x_i(S') - x_i(S) \quad . \tag{2}$$

Matrices $\widehat{\mathbf{Q}}_1^{(x)}$, $\widehat{\mathbf{Q}}_2^{(x)}$ and $\widehat{\mathbf{P}}_2^{(x)}$ represent the 3×3 submatrices of the 6×6 propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$:

$$\mathbf{\Pi}^{(x)}(R,S) = \begin{pmatrix} \widehat{\mathbf{Q}}_1^{(x)}(R,S) & \widehat{\mathbf{Q}}_2^{(x)}(R,S) \\ \widehat{\mathbf{P}}_1^{(x)}(R,S) & \widehat{\mathbf{P}}_2^{(x)}(R,S) \end{pmatrix} \quad . \tag{3}$$

Symbols $p_i(S)$ and $p_i(R)$ denote the Cartesian components of the slowness vectors at points S and R, respectively, and $T(R,S)$ the two-point travel time from S to R, computed along the reference ray.

3. TRANSFORMATION OF THE 6 × 6 RAY-PROPAGATOR MATRICES

In this section, we present the transformation equation for the 6×6 ray-propagator matrices. This transformation equation gives the relation between the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates x_i, and the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray-centred coordinates q_i with $q_3=\tau$, computed by dynamic ray tracing along any reference ray Ω. The ray-propagator matrix plays a basic role in the paraxial ray methods, and can be used to determine the travel time and certain other ray quantities not only along the ray, but also in its quadratic (paraxial) vicinity. Once the ray propagator matrix is known along the reference ray, the solutions of the dynamic ray tracing can be computed by simple matrix multiplication. It has certain powerful properties: it is symplectic, it satisfies the chain rule, its determinant equals 1 along the whole ray (Liouville's theorem), etc. The transformation relation between the 6×6 ray propagator matrices $\mathbf{\Pi}^{(q)}(R,S)$ and $\mathbf{\Pi}^{(x)}(R,S)$ reads (*Červený, 2001, Eq. 4.3.38*):

$$\mathbf{\Pi}^{(q)}(R,S) = \begin{pmatrix} \widehat{\overline{\mathbf{H}}}{}^{(q)}(R) & \widehat{\mathbf{0}} \\ \widehat{\mathbf{U}}_2(R) & \widehat{\mathbf{H}}^{(q)T}(R) \end{pmatrix} \mathbf{\Pi}^{(x)}(R,S) \begin{pmatrix} \widehat{\mathbf{H}}^{(q)}(S) & \widehat{\mathbf{0}} \\ \widehat{\mathbf{U}}_1(S) & \widehat{\overline{\mathbf{H}}}{}^{(q)T}(S) \end{pmatrix} \ , \tag{4}$$

where

$$\widehat{\mathbf{U}}_1 = -\widehat{\overline{\mathbf{H}}}{}^{(q)T}\widehat{\mathbf{F}} \ , \quad \widehat{\mathbf{U}}_2 = \widehat{\mathbf{F}}\widehat{\overline{\mathbf{H}}}{}^{(q)} \quad . \tag{5}$$

Here $\widehat{\mathbf{H}}^{(q)}$ and $\widehat{\overline{\mathbf{H}}}{}^{(q)}$ denote the 3×3 transformation matrices from coordinates q_m to x_i and vice versa, with elements $H_{im}^{(q)} = \partial x_i/\partial q_m$ and $\overline{H}_{mi}^{(q)} = \partial q_m/\partial x_i$. Matrices $\widehat{\mathbf{H}}^{(q)}$ and $\widehat{\overline{\mathbf{H}}}{}^{(q)}$ are always invertible and mutually inverse, $\widehat{\mathbf{H}}^{(q)-1} = \widehat{\overline{\mathbf{H}}}{}^{(q)}$. Symbol $\widehat{\overline{\mathbf{H}}}{}^{(q)}$ is only a matter of notation. The elements F_{mn} of the 3×3 symmetric matrix $\widehat{\mathbf{F}}$ are given by relation

$$F_{mn} = p_i \partial^2 x_i/\partial q_m \partial q_n \quad . \tag{6}$$

The 3×3 matrix $\mathbf{F}$ was introduced by *Klimeš (1994, Eq. 34)*.

Using Eq. (5), it is not difficult to show that Eq. (4) can also be expressed in the reverse form:

$$\mathbf{\Pi}^{(x)}(R,S) = \begin{pmatrix} \widehat{\mathbf{H}}^{(q)}(R) & \widehat{\mathbf{0}} \\ \widehat{\mathbf{U}}_1(R) & \widehat{\overline{\mathbf{H}}}{}^{(q)T}(R) \end{pmatrix} \mathbf{\Pi}^{(q)}(R,S) \begin{pmatrix} \widehat{\overline{\mathbf{H}}}{}^{(q)}(S) & \widehat{\mathbf{0}} \\ \widehat{\mathbf{U}}_2(S) & \widehat{\mathbf{H}}^{(q)T}(S) \end{pmatrix} \quad . \tag{7}$$

As the basic transformation equation (4) is expressed in matrix form, we use the matrix notation systematically. Only at the end of the section, do we express the results also in component form.

We again use the standard notation for the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ as for $\mathbf{\Pi}^{(x)}(R,S)$ in Eq. (3):

$$\mathbf{\Pi}^{(q)}(R,S) = \begin{pmatrix} \widehat{\mathbf{Q}}_1^{(q)}(R,S) & \widehat{\mathbf{Q}}_2^{(q)}(R,S) \\ \widehat{\mathbf{P}}_1^{(q)}(R,S) & \widehat{\mathbf{P}}_2^{(q)}(R,S) \end{pmatrix} \quad . \tag{8}$$

Using Eqs (3) and (8) we can express Eq. (7) in terms of the 3 × 3 submatrices of the 6 × 6 ray propagator matrices $\mathbf{\Pi}^{(x)}(R,S)$ and $\mathbf{\Pi}^{(q)}(R,S)$:

$$\begin{aligned} &\begin{pmatrix} \widehat{\mathbf{Q}}_1^{(x)}(R,S) & \widehat{\mathbf{Q}}_2^{(x)}(R,S) \\ \widehat{\mathbf{P}}_1^{(x)}(R,S) & \widehat{\mathbf{P}}_2^{(x)}(R,S) \end{pmatrix} \\ &= \begin{pmatrix} \widehat{\mathbf{H}}^{(q)}(R)\widehat{\mathbf{Q}}_1^{(q)}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(S) & \widehat{\mathbf{H}}^{(q)}(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{H}}^{(q)T}(S) \\ \widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\widehat{\mathbf{P}}_1^{(q)}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(S) & \widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\widehat{\mathbf{P}}_2^{(q)}(R,S)\widehat{\mathbf{H}}^{(q)T}(S) \end{pmatrix} \\ &+ \begin{pmatrix} \widehat{\mathbf{H}}^{(q)}(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{U}}_2(S) & \widehat{\mathbf{0}} \\ \widehat{\mathbf{W}}^{(q)}(R,S) & \widehat{\mathbf{U}}_1(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{H}}^{(q)T}(S) \end{pmatrix} \quad , \end{aligned} \tag{9}$$

where the 3 × 3 matrix $\widehat{\mathbf{W}}^{(q)}(R,S)$ is given by the relation:

$$\begin{aligned} \widehat{\mathbf{W}}^{(q)}(R,S) &= \widehat{\mathbf{U}}_1(R)\widehat{\mathbf{Q}}_1^{(q)}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(S) \\ &\quad + \widehat{\mathbf{U}}_1(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{U}}_2(S) + \widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\widehat{\mathbf{P}}_2^{(q)}(R,S)\widehat{\mathbf{U}}_2(S) \quad . \end{aligned} \tag{10}$$

This yields the final relations between the 3×3 submatrices of the 6×6 ray propagator matrices in global Cartesian and ray-centred coordinates:

$$\begin{aligned} \widehat{\mathbf{Q}}_1^{(x)}(R,S) &= \widehat{\mathbf{H}}^{(q)}(R)\big[\widehat{\mathbf{Q}}_1^{(q)}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(S) + \widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{U}}_2(S)\big] \quad , \\ \widehat{\mathbf{Q}}_2^{(x)}(R,S) &= \widehat{\mathbf{H}}^{(q)}(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\widehat{\mathbf{H}}^{(q)T}(S) \quad , \\ \widehat{\mathbf{P}}_1^{(x)}(R,S) &= \widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\widehat{\mathbf{P}}_1^{(q)}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(S) + \widehat{\mathbf{W}}^{(q)}(R,S) \quad , \\ \widehat{\mathbf{P}}_2^{(x)}(R,S) &= \big[\widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\widehat{\mathbf{P}}_2^{(q)}(R,S) + \widehat{\mathbf{U}}_1(R)\widehat{\mathbf{Q}}_2^{(q)}(R,S)\big]\widehat{\mathbf{H}}^{(q)T}(S) \quad . \end{aligned} \tag{11}$$

In the transformation of two-point paraxial travel times, however, we do not need to know the transformation of the 3 × 3 submatrices $\widehat{\mathbf{Q}}_1^{(x)}(R,S)$, $\widehat{\mathbf{Q}}_2^{(x)}(R,S)$, $\widehat{\mathbf{P}}_1^{(x)}(R,S)$ and $\widehat{\mathbf{P}}_2^{(x)}(R,S)$, but of the 3×3 matrix $\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)$, and of the two 3×3 matrix products $\widehat{\mathbf{P}}_2^{(x)}(R,S)\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)$ and $\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)\widehat{\mathbf{Q}}_1^{(x)}(R,S)$, see Eq. (1). These quantities can be simply obtained from Eq. (11):

$$\begin{aligned} \widehat{\mathbf{Q}}_2^{(x)-1}(R,S) &= \widehat{\widehat{\mathbf{H}}}^{(q)T}(S)\widehat{\mathbf{Q}}_2^{(q)-1}(R,S)\widehat{\widehat{\mathbf{H}}}^{(q)}(R) \quad , \\ \widehat{\mathbf{P}}_2^{(x)}(R,S)\widehat{\mathbf{Q}}_2^{(x)-1}(R,S) &= \widehat{\widehat{\mathbf{H}}}^{(q)T}(R)\big[\widehat{\mathbf{P}}_2^{(q)}(R,S)\widehat{\mathbf{Q}}_2^{(q)-1}(R,S) - \widehat{\mathbf{F}}(R)\big]\widehat{\widehat{\mathbf{H}}}^{(q)}(R) \quad , \\ \widehat{\mathbf{Q}}_2^{(x)-1}(R,S)\widehat{\mathbf{Q}}_1^{(x)}(R,S) &= \widehat{\widehat{\mathbf{H}}}^{(q)T}(S)\big[\widehat{\mathbf{Q}}_2^{(q)-1}(R,S)\widehat{\mathbf{Q}}_1^{(q)}(R,S) + \widehat{\mathbf{F}}(S)\big]\widehat{\widehat{\mathbf{H}}}^{(q)}(S) \quad . \end{aligned} \tag{12}$$

It should be noted that Eqs (12) do not contain matrix $\widehat{\mathbf{W}}^{(q)}(R,S)$ at all. Consequently, we shall not need it in the further treatment.

Equations (12) are quite general and compact. If we wish to make them more specific, we insert the known expressions for $\widehat{\mathbf{H}}^{(q)}$ and $\widehat{\widehat{\mathbf{H}}}^{(q)}$. They are derived in *Červený et al. (2012)*:

$$\widehat{\mathbf{H}}^{(q)} = (\mathbf{e}_1 \;\; \mathbf{e}_2 \;\; \mathbf{e}_3 = \mathbf{u}) \ , \tag{13}$$

$$\widehat{\widehat{\mathbf{H}}}^{(q)} = (\mathbf{f}_1 \;\; \mathbf{f}_2 \;\; \mathbf{f}_3 = \mathbf{p})^T \ . \tag{14}$$

Here $\mathbf{e}_i$ are contravariant basis vectors and $\mathbf{f}_i$ covariant basis vectors of the ray-centred coordinate system, expressed in Cartesian coordinates. Further, $\mathbf{p}$ is the slowness vector and $\mathbf{u}$ the ray-velocity vector, known from ray tracing. Vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1$ and $\mathbf{f}_2$ are strictly defined in *Červený et al. (2012)*, where also their evaluation is described.

Equations (13) and (14) can be expressed in a more concise form as follows:

$$\widehat{\mathbf{H}}^{(q)} = (\boldsymbol{\mathcal{E}}, \mathbf{u}) \ , \qquad \widehat{\widehat{\mathbf{H}}}^{(q)} = (\boldsymbol{\mathcal{F}}, \mathbf{p})^T \ , \tag{15}$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{F}}$ are 3×2 matrices:

$$\boldsymbol{\mathcal{E}} = (\mathbf{e}_1, \mathbf{e}_2) \ , \qquad \boldsymbol{\mathcal{F}} = (\mathbf{f}_1, \mathbf{f}_2) \ , \tag{16}$$

On the right-hand sides of Eqs (12), we further need to know the 3×3 submatrices of the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray-centred coordinates. In this matrix, the third and sixth columns and lines are extremely simple. All elements in these columns and lines vanish, with the exception of three: $\Pi^{(q)}_{33}(R,S) = \Pi^{(q)}_{66}(R,S) = 1$, $\Pi^{(q)}_{36}(R,S) = T(R,S)$ (see *Klimeš, 1994*). Consequently, we can express the 3×3 submatrices $\widehat{\mathbf{Q}}^{(q)}_1(R,S)$, $\widehat{\mathbf{Q}}^{(q)}_2(R,S)$, $\widehat{\mathbf{P}}^{(q)}_1(R,S)$ and $\widehat{\mathbf{P}}^{(q)}_2(R,S)$ of the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in terms of 2×2 matrices $\mathbf{Q}^{(q)}_1(R,S)$, $\mathbf{Q}^{(q)}_2(R,S)$, $\mathbf{P}^{(q)}_1(R,S)$ and $\mathbf{P}^{(q)}_2(R,S)$, in the following way:

$$\widehat{\mathbf{Q}}^{(q)}_1(R,S) = \begin{pmatrix} \mathbf{Q}^{(q)}_1(R,S) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \ , \quad \widehat{\mathbf{P}}^{(q)}_1(R,S) = \begin{pmatrix} \mathbf{P}^{(q)}_1(R,S) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{pmatrix} \ , \tag{17}$$

and

$$\widehat{\mathbf{Q}}^{(q)}_2(R,S) = \begin{pmatrix} \mathbf{Q}^{(q)}_2(R,S) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & T(R,S) \end{pmatrix} \ , \quad \widehat{\mathbf{P}}^{(q)}_2(R,S) = \begin{pmatrix} \mathbf{P}^{(q)}_2(R,S) & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \ . \tag{18}$$

Here $T(R,S)$ is again the travel time from point S to point R, situated along reference ray Ω. Thus, it is not necessary to compute the complete 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$: it is sufficient to compute only the 4×4 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$, composed of four 2×2 submatrices $\mathbf{Q}^{(q)}_1(R,S)$, $\mathbf{Q}^{(q)}_2(R,S)$, $\mathbf{P}^{(q)}_1(R,S)$, $\mathbf{P}^{(q)}_2(R,S)$. This considerably simplifies the dynamic ray tracing in ray-centred coordinates. Remark concerning the notation: We do not introduce a new symbol for the 4×4 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray-centred coordinates to distinguish it from the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ as we always specify the dimensions of the ray-propagator matrices used in the following text. Using Eqs (12), (14) and (16)–(18), we obtain

$$
\begin{aligned}
\widehat{\mathbf{Q}}_2^{(x)-1}(R,S) &= \boldsymbol{\mathcal{F}}(S)\mathbf{Q}_2^{(q)-1}(R,S)\boldsymbol{\mathcal{F}}^T(R) + T^{-1}(R,S)\mathbf{p}(S)\mathbf{p}^T(R) \quad , \\
\widehat{\mathbf{P}}_2^{(x)}(R,S)\widehat{\mathbf{Q}}_2^{(x)-1}(R,S) &= \boldsymbol{\mathcal{F}}(S)\mathbf{P}_2^{(q)}(R,S)\mathbf{Q}_2^{(q)-1}(R,S)\boldsymbol{\mathcal{F}}^T(R) \\
&\quad + T^{-1}(R,S)\mathbf{p}(R)\mathbf{p}^T(R) + \widehat{\boldsymbol{\Phi}}(R) \quad , \\
\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)\widehat{\mathbf{Q}}_1^{(x)}(R,S) &= \boldsymbol{\mathcal{F}}(S)\mathbf{Q}_2^{(q)-1}(R,S)\mathbf{Q}_1^{(q)}(R,S)\boldsymbol{\mathcal{F}}^T(S) \\
&\quad + T^{-1}(R,S)\mathbf{p}(S)\mathbf{p}^T(S) - \widehat{\boldsymbol{\Phi}}(S) \quad .
\end{aligned}
\tag{19}
$$

The 3×3 matrix $\widehat{\boldsymbol{\Phi}}$ is given by the relation

$$
\widehat{\boldsymbol{\Phi}} = -\widehat{\widehat{\mathbf{H}}}^{(q)T}\widehat{\mathbf{F}}\widehat{\widehat{\mathbf{H}}}^{(q)} \quad , \tag{20}
$$

where the symmetric 3×3 matrix $\widehat{\mathbf{F}}$ is given by the relation

$$
-\widehat{\mathbf{F}} = \begin{pmatrix} 0 & 0 & \mathbf{e}_1^T\boldsymbol{\eta} \\ 0 & 0 & \mathbf{e}_2^T\boldsymbol{\eta} \\ \mathbf{e}_1^T\boldsymbol{\eta} & \mathbf{e}_2^T\boldsymbol{\eta} & \mathbf{u}^T\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \boldsymbol{\mathcal{E}}^T\boldsymbol{\eta} \\ \boldsymbol{\eta}^T\boldsymbol{\mathcal{E}} & \mathbf{u}^T\boldsymbol{\eta} \end{pmatrix} \quad . \tag{21}
$$

Here $\mathbf{0}$ denotes the 2×2 null matrix. Equation (21) follows from Eq. (6), see *Klimeš (1994, Eq. 36)*, *Červený (2001, Eq. 4.2.76)*. Equations (20) and (21) are valid at any point of the ray, including points S and R. For these reasons, we do not include arguments with the individual symbols. Inserting Eqs (21) and (15) into Eq. (20), we obtain:

$$
\begin{aligned}
\widehat{\boldsymbol{\Phi}} &= (\boldsymbol{\mathcal{F}}, \mathbf{p}) \begin{pmatrix} \mathbf{0} & \boldsymbol{\mathcal{E}}^T\boldsymbol{\eta} \\ \boldsymbol{\eta}^T\boldsymbol{\mathcal{E}} & \mathbf{u}^T\boldsymbol{\eta} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mathcal{F}}^T \\ \mathbf{p}^T \end{pmatrix} \\
&= \mathbf{p}\boldsymbol{\eta}^T\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}^T + \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{E}}^T\boldsymbol{\eta}\mathbf{p}^T + \mathbf{p}\mathbf{u}^T\boldsymbol{\eta}\mathbf{p}^T \quad .
\end{aligned}
\tag{22}
$$

To determine $\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}^T$ and $\boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{E}}^T$, we use the relation $\widehat{\mathbf{H}}^{(q)}\widehat{\widehat{\mathbf{H}}}^{(q)} = \widehat{\mathbf{I}}$, where $\widehat{\mathbf{I}}$ is the 3×3 identity matrix, and obtain

$$
\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{F}}^T = \widehat{\mathbf{I}} - \mathbf{u}\mathbf{p}^T \quad , \qquad \boldsymbol{\mathcal{F}}\boldsymbol{\mathcal{E}}^T = \widehat{\mathbf{I}} - \mathbf{p}\mathbf{u}^T \quad . \tag{23}
$$

Inserting Eq. (23) into Eq. (22) we obtain

$$
\widehat{\boldsymbol{\Phi}} = \mathbf{p}\boldsymbol{\eta}^T(\widehat{\mathbf{I}} - \mathbf{u}\mathbf{p}^T) + (\widehat{\mathbf{I}} - \mathbf{p}\mathbf{u}^T)\boldsymbol{\eta}\mathbf{p}^T + \mathbf{p}\mathbf{p}^T(\mathbf{u}^T\boldsymbol{\eta}) \quad . \tag{24}
$$

This finally yields

$$
\widehat{\boldsymbol{\Phi}} = \mathbf{p}\boldsymbol{\eta}^T + \boldsymbol{\eta}\mathbf{p}^T - (\mathbf{u}^T\boldsymbol{\eta})(\mathbf{p}\mathbf{p}^T) \quad . \tag{25}
$$

This equation does not depend on the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1$ and $\mathbf{f}_2$, but only on vectors $\mathbf{p}, \mathbf{u}$ and $\boldsymbol{\eta}$, known from ray tracing.

Equations (19), together with Eq. (25), yield the final expressions for the 3×3 matrices $\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)$, $\widehat{\mathbf{P}}_2^{(x)}(R,S)\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)$ and $\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)\widehat{\mathbf{Q}}_1^{(x)}(R,S)$ we have been looking for. We can also simply express them in component forms as follows:

$$
\begin{aligned}
(\widehat{\mathbf{Q}}_2^{(x)-1}(R,S))_{ij} &= f_{Mi}(S)(\mathbf{Q}_2^{(q)-1}(R,S))_{MN} f_{Nj}(R) + T^{-1}(R,S) p_i(S) p_j(R) \quad , \\
(\widehat{\mathbf{P}}_2^{(x)}(R,S)\widehat{\mathbf{Q}}_2^{(x)-1}(R,S))_{ij} &= f_{Mi}(R)(\mathbf{P}_2^{(q)}(R,S)\mathbf{Q}_2^{(q)-1}(R,S))_{MN} f_{Nj}(R) \\
&\quad + T^{-1}(R,S) p_i(R) p_j(R) + \Phi_{ij}(R) \quad , \\
(\widehat{\mathbf{Q}}_2^{(x)-1}(R,S)\widehat{\mathbf{Q}}_1^{(x)}(R,S))_{ij} &= f_{Mi}(S)(\mathbf{Q}_2^{(q)-1}(R,S)\mathbf{Q}_1^{(q)}(R,S))_{MN} f_{Nj}(S) \\
&\quad + T^{-1}(R,S) p_i(S) p_j(S) - \Phi_{ij}(S) \quad ,
\end{aligned}
\tag{26}
$$

where

$$
\Phi_{ij} = p_i \eta_j + \eta_i p_j - (u_k \eta_k) p_i p_j \quad . \tag{27}
$$

4. DERIVATION OF TWO-POINT PARAXIAL TRAVEL TIME FORMULA IN RAY-CENTRED COORDINATES FROM AN ANALOGOUS FORMULA IN GLOBAL CARTESIAN COORDINATES

The formula for the two-point paraxial travel time $T(R',S')$ expressed in terms of the 4×4 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ computed by dynamic ray tracing in ray-centred coordinates q_i, can be simply determined from the formula (1) for the two-point paraxial travel time $T(R',S')$ expressed in terms of the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ computed by dynamic ray tracing in global Cartesian coordinates. For this purpose, we insert Eqs (26) into Eq. (1). The equation simplifies considerably as all terms with $T^{-1}(R,S)$ are mutually eliminated. Finally, we obtain:

$$
\begin{aligned}
T(R',S') = T(R,S) &+ \delta x_i^R p_i(R) - \delta x_i^S p_i(S) \\
&+ \tfrac{1}{2}\delta x_i^R [f_{Mi}(R)(\mathbf{P}_2^{(q)}\mathbf{Q}_2^{(q)-1})_{MN} f_{Nj}(R) + \Phi_{ij}(R)]\delta x_j^R \\
&+ \tfrac{1}{2}\delta x_i^S [f_{Mi}(S)(\mathbf{Q}_2^{(q)-1}\mathbf{Q}_1^{(q)})_{MN} f_{Nj}(S) - \Phi_{ij}(S)]\delta x_j^S \\
&- \delta x_i^S [f_{Mi}(S)(\mathbf{Q}_2^{(q)-1})_{MN} f_{Nj}(R)]\delta x_j^R \quad .
\end{aligned}
\tag{28}
$$

We can also use Eqs (26) in the opposite way, to obtain Eq. (1) from Eq. (28). Equation (28) is exactly the same as Eq. (40) in *Červený et al. (2012)*.

5. CONCLUSION

Both Eqs (1) and (28) are known from the paper by *Červený et al. (2012)*. They were derived quite independently there, using quite different methods. Here we have proved that both equations are mutually related by the transformation equation (4) between the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates x_i and the 6×6 ray-propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray-centred coordinates, computed by dynamic ray tracing along the reference ray Ω, known from *Červený (2001, Sec. 4.3.6/3)*.

Acknowledgements: I would like to express my gratitude to Ivan Pšenčík, Einar Iversen and two anonymous referees for their stimulating comments. The research has been supported by the Grant Agency of the Czech Republic under Contracts P210/10/0736 and P210/11/0117, by the Ministry of Education, Youth and Sports of the Czech Republic within Research Project MSM0021620860, and by the Consortium Project "Seismic Waves in Complex 3-D Structures".

References

Červený V., 2001. *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge, U.K.

Červený V., Iversen E. and Pšenčík I., 2012. Two-point paraxial traveltimes in an inhomogeneous anisotropic medium. *Geophys. J. Int.*, **189**, 1597–1610.

Červený V. and Klimeš L., 2010. Transformation relations for second-order derivatives of travel time in anisotropic media. *Stud. Geophys. Geod.*, **54**, 257–267.

Hamilton W.R., 1837. Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144.

Klimeš L., 1994. Transformations for dynamic ray tracing in anisotropic media. *Wave Motion*, **20**, 261–272.

Klimeš L., 2009. Relation between the propagator matrix of geodesic deviation and second-order derivatives of characteristic function. In: *Seismic Waves in Complex 3-D Structures, Report 19.* Dep. Geophys., Charles Univ., Prague, 103–114 (online at "http://sw3d.cz").