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Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function

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In the Finsler geometry, which is a generalization of the Riemann geometry, the metric tensor also depends on the direction of propagation. The basics of the Finsler geometry were formulated by William Rowan Hamilton in 1832. Hamilton's formulation is based on the first-order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between two points. The characteristic function and geodesics together with the geodesic deviation in the Finsler space can be calculated efficiently by Hamilton's method. The Hamiltonian equations of geodesic deviation are considerably simpler than the Riemannian or Finslerian equations of geodesic deviation. The linear ordinary differential equations of geodesic deviation may serve to calculate geodesic deviations, amplitudes of waves and the second-order spatial derivatives of the characteristic function or action. The propagator matrix of geodesic deviation contains all the linearly independent solutions of the equations of geodesic deviation.

In this paper, we use the Hamiltonian formulation to derive the relation between the propagator matrix of geodesic deviation and the second-order spatial derivatives of the characteristic function in the Finsler geometry. We assume that the Hamiltonian function is a positively homogeneous function of the second degree with respect to the spatial gradient of the characteristic function, which corresponds to the Riemannian or Finslerian equations of geodesics and of geodesic deviation. The derived equations, which represent the main result of this paper, are applicable to the Finsler geometry, the Riemann geometry, and their various applications such as general relativity or the high-frequency approximations of wave propagation.

1. Introduction

In the Riemann geometry, the metric tensor depends on coordinates. In the Finsler geometry, which is a generalization of the Riemann geometry, the metric tensor also depends on the direction of propagation. All equations applicable to the Finsler space are equally applicable to the Riemann space. Seismic, electromagnetic and other waves in the high-frequency approximation often propagate according to the Finsler geometry. The characteristic function and the propagator matrix of geodesic deviation have found many important applications in wave propagation, see [1].

The basics of the Finsler geometry were formulated by Sir William Rowan Hamilton in 1832, see [2]. Hamilton's formulation is based on the first-order partial differential Hamilton–Jacobi equations for the characteristic function which represents the distance between two points. On the other hand, Finsler's [3] formulation is based on differential geometry. The relation between Finsler's formulation and Hamilton's formulation is well known, see [4, Section 5.1], but the simplicity and power of Hamilton's formulation are underestimated too much in the

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Finsler geometry. The characteristic function and geodesics in the Finsler space can be calculated efficiently by Hamilton's method. The Hamiltonian formulation simplifies many equations of the Riemann geometry. In the Hamiltonian formulation, the description of geodesics and of geodesic deviation in the Finsler geometry is no more difficult than in the Riemann geometry.

The non-linear ordinary differential equations of geodesics (equations of rays) may serve to calculate geodesics, the characteristic function (point-to-point distance, two-point travel time) with its first-order spatial derivatives, or the action (distance for general initial conditions, travel time for general initial conditions) with its first-order spatial derivatives. The linear ordinary differential equations of geodesic deviation may serve to calculate geodesic deviations, amplitudes of waves, the second-order spatial derivatives of the characteristic function, or the second-order spatial derivatives of the action. Geodesic perturbations, higher-order geodesic deviations, and the perturbation derivatives and higher-order spatial derivatives of the characteristic function or of the action can be calculated by quadratures along geodesics, see [5,6].

In this paper, we use the Hamiltonian formulation to derive the relation between the propagator matrix of geodesic deviation and the second-order spatial derivatives of the characteristic function in the Finsler geometry. The derived equations, which represent the main result of this paper, are applicable to the Finsler geometry, to the Riemann geometry, and to their various applications such as general relativity or the high-frequency approximations of wave propagation.

We assume that the Hamiltonian function is a positively homogeneous function of the second degree with respect to the spatial gradient of the characteristic function, which corresponds to the Riemannian or Finslerian equations of geodesics and of geodesic deviation. The equations derived in this paper were generalized to an arbitrary Hamiltonian function by Klimeš, see [7].

2. Hamiltonian formulation of the Finsler geometry

2.1. *Homogeneous Hamiltonian function of the second degree and the metric tensor*

We consider a smooth manifold (differentiable manifold), and coordinates x^i of its coordinate chart. At each point x^i, we have the tangent space containing contravariant vectors y^i and the cotangent space containing covariant vectors y_i such as the gradients of functions. We consider Hamiltonian function $H(x^i, y_j)$, which is a real-valued function of coordinates x^i and of covariant vector y_j from the cotangent space at point x^i, and which is differentiable within its definition domain. Hamilton [2] assumed the Hamiltonian function to be a positively homogeneous function of the first degree with respect to y_j, but his theory is applicable to general Hamiltonian functions as well.

The Riemannian or Finslerian equations of geodesics and of geodesic deviation correspond to the positively homogeneous Hamiltonian function $H(x^i, y_j)$ of the second degree with respect to y_j. In this paper, we thus assume that the Hamiltonian function $H(x^i, y_j)$ is a positively homogeneous function of the second degree with respect to y_j, especially in Section 3.2 and the whole Section 4. Note that in the Riemann geometry, $H(x^m, y_n) = \frac{1}{2} y_i\, g^{ij}(x^m)\, y_j$, where $g^{ij}(x^m)$ are the contravariant components of the Riemann metric tensor.

Hamilton [2] called the subset of the unit vectors y_j in the cotangent space at point x^i, defined by equation

$$H(x^i, y_j) = \frac{1}{2}, \tag{1}$$

the *surface of components of normal slowness*. Now it is often called the *phase-slowness surface* or briefly the *slowness surface*, sometimes the *index surface*. In the Finsler geometry, it is referred to as the *figuratrix*.

Contravariant vector y^i corresponding to covariant vector y_j is defined by relation

$$y^i = \frac{\partial H}{\partial y_i}(x^m, y_n). \tag{2}$$

For example, if covariant vector y_i represents the slowness vector, the corresponding contravariant vector y^i represents the velocity vector.

The contravariant components of the metric tensor are defined by equation

$$g^{ij}(x^m, y_n) = \frac{\partial^2 H}{\partial y_i \partial y_j}(x^m, y_n), \tag{3}$$

see [8,Equation (1.5.4)].

2.2. Action, the characteristic function and the Hamilton–Jacobi equations

The Hamilton–Jacobi equation is a partial differential equation of the first-order. The Hamilton–Jacobi equation for *action* (*distance for general initial conditions*, *travel time for general initial conditions*) $S(x^m)$ reads

$$H\big(x^i, \tfrac{\partial S}{\partial x^j}(x^m)\big) = \frac{1}{2}. \tag{4}$$

Hamilton [2] also defined the *characteristic function* (*point-to-point distance*, *two-point travel time*)

$$V(x^m, \tilde{x}^n) \tag{5}$$

from the point of coordinates $\tilde{x}^n$ to the point of coordinates x^m. The characteristic function satisfies the Hamilton–Jacobi equations

$$H\big(x^m, \tfrac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\big) = \frac{1}{2} \tag{6}$$

and

$$H\big(\tilde{x}^m, -\tfrac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\big) = \frac{1}{2}, \tag{7}$$

see [2,Equation (C)]. Note that one of Equations (6) and (7) serves as the initial conditions to the other. The Hamilton–Jacobi equations express the requirement that the gradient of the action or of the characteristic function is unit, see the definition (1) of unit covariant vectors.

2.3. Equations of geodesics

Hamilton's equations (*equations of geodesics*) read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x^i = \frac{\partial H}{\partial y_i}(x^m, y_n), \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}y_i = -\frac{\partial H}{\partial x^i}(x^m, y_n). \tag{9}$$

Hamilton [2] referred to these equations as the *general equations of rays*. Today, they are also called the *equations of rays* or the *ray-tracing equations*. Hamilton's equations (8) and (9) can simply be derived by differentiating the Hamilton–Jacobi equation (4) or (6) with respect to coordinate x^j. The meaning of the independent parameter τ along the geodesic and the sensitivity of the geodesic to the initial conditions depend on the form of the Hamiltonian function. Covariant vector y_i in (8) and (9), which represents the first-order partial derivatives of the characteristic function (or action) with respect to spatial coordinates, was called the *normal slowness* by Hamilton, see [2]. Now it is usually called the *slowness vector*. Note that we may obtain analogous Hamilton's equations for initial point $\tilde{x}^j$ if we differentiate the second Hamilton–Jacobi equation (7) with respect to coordinates $\tilde{x}^j$, see [7,Equations (9) and (10)].

For the positively homogeneous Hamiltonian function H of the second degree, Hamilton's equations (8) and (9) are equivalent to the Riemannian or Finslerian equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}x^i = g^{ij}\, y_j, \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}y_i = \Gamma^j_{ik}\, y_j\, g^{kl}\, y_l \tag{11}$$

of geodesics, see [8,Equations (1.5.8) and (2.2.5)]. Here Γ^j_{ik} represents the Christoffel symbols of the first kind, see [8,Equations (2.2.3) and (2.2.7)]. For a general Hamiltonian function, Equations (8)–(9) and (10)–(11) differ just by the independent parameter τ along the geodesic and by the sensitivity of the geodesic to the initial conditions.

2.4. Equations of geodesic deviation

We define vectors

$$X^i = \frac{\partial x^i}{\partial \gamma} \tag{12}$$

and

$$Y_i = \frac{\partial y_i}{\partial \gamma} \tag{13}$$

representing the geodesic deviation corresponding to some parameter γ parametrizing the initial conditions for the geodesics. Since the geodesics are parametrized by independent parameters τ and γ, derivatives $\frac{\mathrm{d}}{\mathrm{d}\tau}$ and $\frac{\partial}{\partial\gamma}$ commute. The equations for X^i and Y_i are then obtained by differentiating Hamilton's equations (8) and (9) with respect to γ. The resulting *Hamiltonian equations of geodesic deviation* (*paraxial ray equations*, *dynamic ray tracing equations*) derived by Červený [9] read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X^i = \quad H^{,i}_{,j}\, X^j + H^{,ij}\, Y_j, \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y_i = -H_{,ij}\, X^j - H^{,j}_{,i}\, Y_j, \tag{15}$$

where

$$H_{,ij} = \frac{\partial^2 H}{\partial x^i \partial x^j}(x^m, y_n), \tag{16}$$

$$H^{,i}_{,j} = \frac{\partial^2 H}{\partial y_i \partial x^j}(x^m, y_n), \tag{17}$$

$$H^{,ij} = \frac{\partial^2 H}{\partial y_i \partial y_j}(x^m, y_n). \tag{18}$$

For the positively homogeneous Hamiltonian function H of the second degree, the Hamiltonian equations (14) and (15) of geodesic deviation are equivalent to the Riemannian or Finslerian equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}X^i + \Gamma^i_{js}\, y^s\, X^j = g^{ij}\, \tilde{Y}_j, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{Y}_i - \Gamma^j_{is}\, y^s\, \tilde{Y}_j = -R_{irsj}\, y^r\, y^s\, X^j \tag{20}$$

of geodesic deviation, see [8, Equation (4.4.16)]. Here, R_{isrj} represents any one of the relevant curvature tensors, see [8,Equations (4.2.12) or (4.2.15)], and

$$\tilde{Y}_i = Y_i - \Gamma^s_{ij}\, y_s\, X^j. \tag{21}$$

For a general Hamiltonian function, Equations (14)–(15) and (19)–(20) differ.

2.5. *Propagator matrix of geodesic deviation*

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a is defined by equation

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \dfrac{\partial x^i}{\partial \tilde{x}^j} & \dfrac{\partial x^i}{\partial \tilde{y}_j} \\ \dfrac{\partial y_i}{\partial \tilde{x}^j} & \dfrac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix}, \tag{22}$$

where the derivatives are taken at fixed parameter τ along geodesics. The propagator matrix of geodesic deviation is symplectic [1] and obeys the chain rule,

$$\mathbf{\Pi}(x^a, \tilde{x}^c) = \mathbf{\Pi}(x^a, \overline{x}^b)\, \mathbf{\Pi}(\overline{x}^c, \tilde{x}^d), \tag{23}$$

where $\tilde{x}^d$, $\overline{x}^c$ and x^a are the coordinates of three points along a geodesic.

The propagator matrix of geodesic deviation contains all linearly independent solutions of the equations of geodesic deviation, and may thus be used to calculate the geodesic deviation for any initial conditions.

In the high-frequency approximations of wave propagation, the submatrix $\frac{\partial x^i}{\partial \tilde{y}_j}$ of the propagator matrix of geodesic deviation determines the amplitude of the Green function of the corresponding wave field. The amplitude of the Green function is proportional to expression $V^{\frac{1}{2}} \left| \det \left(\frac{\partial x^i}{\partial \tilde{y}_j} \right) \right|^{-\frac{1}{2}}$. The whole propagator matrix of geodesic deviation may be used to calculate the amplitude for any initial conditions. The relation between the propagator matrix of geodesic deviation and the second-order spatial derivatives of the characteristic function is proposed in Section 3.2.

The Hamiltonian equations (14) and (15) of geodesic deviation for the propagator matrix read

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} H^{,i}_{,j} & H^{,ij} \\ -H_{,ij} & -H^{,j}_{,i} \end{pmatrix} \mathbf{\Pi}(x^a, \tilde{x}^b), \tag{24}$$

with unit initial conditions. Note that it is also possible to define the covariant version of the propagator matrix of geodesic deviation, but we do not need it here.

3. Derivatives of the characteristic function

3.1. *First-order spatial derivatives of the characteristic function*

The first-order spatial derivatives

$$\frac{\partial V}{\partial x^i} = y_i, \tag{25}$$

$$\frac{\partial V}{\partial \tilde{x}^i} = -\tilde{y}_i \tag{26}$$

of the characteristic function result from the solution of Hamilton's equations (8) and (9), see [2].

3.2. *Relation between the propagator matrix of geodesic deviation and the second-order spatial derivatives of the characteristic function*

The linear ordinary differential Hamiltonian equations (24) of geodesic deviation can be used to calculate the propagator matrix (22) of geodesic deviation. The second-order spatial derivatives of the characteristic function can be obtained from the propagator matrix (22) of geodesic deviation.

The relations between the second-order spatial derivatives of the characteristic function and the propagator matrix of geodesic deviation can be derived in the following way: The relations between the second-order spatial derivatives of the characteristic function and the propagator matrix of geodesic deviation in ray-centred coordinates in isotropic media given by Arnaud [10] and Červený, Klimeš & Pšenčík [11] can simply be generalized to the propagator matrix [12, Equation (59)] of geodesic deviation in ray-centred coordinates in anisotropic media, i.e. in the Finsler geometry. These relations can then be transformed to general coordinates using the transformation equations by Klimeš, see [12,Equations (64) and (65)], and Červený and Klimeš, see [13,Equation (36)].

The Finslerian equations of geodesic deviation correspond to a positively homogeneous Hamiltonian function of the second degree. For the positively homogeneous Hamiltonian function of the second degree with respect to the spatial gradient of the characteristic function, the derived unique relations between the second-order spatial derivatives of characteristic function (5) and the propagator matrix (22) of geodesic deviation read

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k}, \tag{27}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k, \tag{28}$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^j}\frac{\partial V}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k}, \tag{29}$$

where Kronecker delta δ_i^k represents the components of the identity matrix. These relations, which represent the main result of this paper, are proved in Section 4.

Only three submatrices of the propagator matrix (22) of geodesic deviation are used in Equations (27)–(29). Note that the fourth submatrix $\partial y_i/\partial \tilde{x}^k$ of matrix (22) carries no additional information; it can be calculated from the three submatrices used in Equations (27)–(29) using the symplectic property of the propagator matrix (22) of geodesic deviation.

4. Proof of the relation between the propagator matrix of geodesic deviation and the second-order spatial derivatives of the characteristic function

We first calculate the limits of Equations (27)–(29) at the initial point of a geodesic, i.e. for point x^a approaching the initial point $\tilde{x}^a$ of the geodesic, and show that Equations (27)–(29) hold at the initial point.

We then assume that Equations (27)–(29) are satisfied at point x^a of the geodesic, calculate the derivatives of Equations (27)–(29) along the geodesic at point x^a, and show that these derivatives hold. In this way, we prove that Equations (27)–(29) hold along the whole geodesic.

4.1. Limit of the relations at the initial point of a geodesic

For small distances between $\tilde{x}^a$ and x^b, the characteristic function can be approximated as [8, Equation (1.1.14)]

$$V(x^b, \tilde{x}^a) = F(\tilde{x}^m, x^n - \tilde{x}^n) + O(V^2), \tag{30}$$

where $F(x^i, y^j)$ is the fundamental function, which specifies the length of the infinitesimally small vectorial distance y^j, and is thus positively homogeneous of the first degree with respect to y^j. The approximate velocity vector along the geodesic from $\tilde{x}^a$ to x^a reads

$$y^i = \frac{x^i - \tilde{x}^i}{F(\tilde{x}^m, x^n - \tilde{x}^n)}. \tag{31}$$

The slowness vector corresponding to this velocity vector is

$$y_i(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial F^2}{\partial y^i}(\tilde{x}^m, y^n), \tag{32}$$

see [8,Equation (1.5.6)], and the corresponding covariant metric tensor is defined by relation

$$g_{ij}(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(\tilde{x}^m, y^n), \tag{33}$$

see [8,Equation (1.3.1)]. Inserting (31) into the right-hand side of (33), and considering the positive homogeneity of the fundamental function $F(x^i, y^j)$ and of its derivatives with respect to y^j, we obtain

$$g_{ij}(\tilde{x}^m, y^n) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(\tilde{x}^m, x^n - \tilde{x}^n). \tag{34}$$

Since (34) contains the square of the fundamental function F, we express approximation (30) as

$$V(x^b, \tilde{x}^a) = \sqrt{F^2(\tilde{x}^m, x^n - \tilde{x}^n)} + O(V^2). \tag{35}$$

We now differentiate approximation (35) once,

$$\frac{\partial V}{\partial x^i}(x^b, \tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial F^2}{\partial y^i}\right](\tilde{x}^m, x^n - \tilde{x}^n) + O(V^1), \tag{36}$$

$$\frac{\partial V}{\partial \tilde{x}^i}(x^b, \tilde{x}^a) = -\left[\frac{1}{2\sqrt{F^2}}\frac{\partial F^2}{\partial y^i}\right](\tilde{x}^m, x^n - \tilde{x}^n) + O(V^1), \tag{37}$$

and twice,

$$\frac{\partial^2 V}{\partial x^i \partial x^j}(x^b, \tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n - \tilde{x}^n) + O(V^0), \tag{38}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j}(x^b, \tilde{x}^a) = -\left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n - \tilde{x}^n) + O(V^0), \tag{39}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j}(x^b, \tilde{x}^a) = \left[\frac{1}{2\sqrt{F^2}}\frac{\partial^2 F^2}{\partial y^i \partial y^j} - \frac{1}{4\left[\sqrt{F^2}\right]^3}\frac{\partial F^2}{\partial y^i}\frac{\partial F^2}{\partial y^j}\right](\tilde{x}^m, x^n - \tilde{x}^n) + O(V^0). \tag{40}$$

Inserting relations (34), (35), (36) and (37) into (38)–(40), we arrive at

$$\frac{\partial^2 V}{\partial x^i \partial x^j}(x^b, \tilde{x}^a) = \frac{1}{V(x^b, \tilde{x}^a)}\left[g_{ij}(\tilde{x}^m, y^n) - \frac{\partial V}{\partial x^i}(x^b, \tilde{x}^a)\frac{\partial V}{\partial x^j}(x^d, \tilde{x}^c)\right] + O(V^0), \tag{41}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j}(x^b, \tilde{x}^a) = \frac{1}{V(x^b, \tilde{x}^a)}\left[-g_{ij}(\tilde{x}^m, y^n) - \frac{\partial V}{\partial \tilde{x}^i}(x^b, \tilde{x}^a)\frac{\partial V}{\partial x^j}(x^d, \tilde{x}^c)\right] + O(V^0), \tag{42}$$

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j}(x^b, \tilde{x}^a) = \frac{1}{V(x^b, \tilde{x}^a)}\left[g_{ij}(\tilde{x}^m, y^n) - \frac{\partial V}{\partial \tilde{x}^i}(x^b, \tilde{x}^a)\frac{\partial V}{\partial \tilde{x}^j}(x^d, \tilde{x}^c)\right] + O(V^0). \tag{43}$$

From the Hamiltonian equations (24) of geodesic deviation, we see that

$$\begin{pmatrix} \frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\ \frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} = \begin{pmatrix} \delta^i_j + \frac{\partial^2 H}{\partial y_i \partial x^j} V & \frac{\partial^2 H}{\partial y_i \partial y_j} V \\ -\frac{\partial^2 H}{\partial x^i \partial x^j} V & \delta^j_i - \frac{\partial^2 H}{\partial x^i \partial y_j} V \end{pmatrix} + O(V^2). \tag{44}$$

Inserting (3) into (44), we express

$$\frac{\partial x^i}{\partial \tilde{y}_j} = V\, g^{ij} + O(V^2). \tag{45}$$

We multiply (41)–(43) by (45) and obtain approximations

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \delta^k_i + O(V^1), \tag{46}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta^k_i + O(V^1), \tag{47}$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial \tilde{x}^j} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^i} \frac{\partial V}{\partial \tilde{x}^j} \right) \frac{\partial x^k}{\partial \tilde{y}_j} = \delta^k_i + O(V^1). \tag{48}$$

From (44) and (46)–(48), we see that Equations (27)–(29) are satisfied for small distances between $\tilde{x}^a$ and x^a with the accuracy of $O(V^1)$, and are thus satisfied for $x^a \to \tilde{x}^a$.

4.2. Differentiating the Hamilton–Jacobi equations

Hereinafter, a subscript following a comma denotes the partial derivative with respect to coordinate x^i, e.g. $H_{,i} = \frac{\partial H}{\partial x^i}$ or $V_{,i} = \frac{\partial V}{\partial x^i}$. A subscript with a tilde following a comma denotes the partial derivative with respect to initial coordinate $\tilde{x}^a$, e.g. $V_{,\tilde{a}} = \frac{\partial V}{\partial \tilde{x}^a}$. A superscript following a comma denotes the partial derivative with respect to slowness vector component y_i, e.g. $H^{,i} = \frac{\partial H}{\partial y_i}$.

We differentiate the Hamilton–Jacobi equation (6) with respect to x^i and obtain equation.

$$H_{,i} + H^{,k} V_{,ki} = 0, \tag{49}$$

see [2,Equations (Q), (I), (K)]. We differentiate the Hamilton–Jacobi equation (6) with respect to $\tilde{x}^a$ and obtain equation

$$H^{,k} V_{,k\tilde{a}} = 0, \tag{50}$$

see [2,Equations (U), (I)].

We differentiate Equation (49) with respect to x^j and obtain equation

$$H_{,ij} + H_{,i}^{,m} V_{,mj} + V_{,im} H_{,j}^{,m} + V_{,im} H^{,mn} V_{,nj} + H^{,k} V_{,kij} = 0. \tag{51}$$

We differentiate Equation (49) with respect to $\tilde{x}^a$, or Equation (50) with respect to x^j, and obtain equation

$$V_{,\tilde{a}m} H_{,j}^{,m} + V_{,\tilde{a}m} H^{,mn} V_{,nj} + H^{,k} V_{,k\tilde{a}j} = 0. \tag{52}$$

We differentiate Equation (50) with respect to $\tilde{x}^b$ and obtain equation

$$V_{,\tilde{a}m} H^{,mn} V_{,n\tilde{b}} + H^{,k} V_{,k\tilde{a}\tilde{b}} = 0. \tag{53}$$

4.3. Derivatives along geodesics

Equation (8) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = H^{,k}\frac{\partial}{\partial x^k}. \tag{54}$$

We apply Equation (54) to V and obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V = H^{,k}V_{,k}. \tag{55}$$

Because of the positive homogeneity of Hamiltonian function H, Equation (55) reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V = 1. \tag{56}$$

Equation (49) with (54) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V_{,i} = -H_{,i}. \tag{57}$$

Equation (50) with (54) yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V_{,\tilde{a}} = 0. \tag{58}$$

Equation (51) with (54) yields the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V_{,ij} = -H_{,ij} - H_{,i}^{,m}V_{,mj} - V_{,im}H_{,j}^{,m} - V_{,im}H^{,mn}V_{,nj} \tag{59}$$

for $V_{,ij}$. Equation (52) with (54) yields equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V_{,\tilde{a}j} = -V_{,\tilde{a}m}H_{,j}^{,m} - V_{,\tilde{a}m}H^{,mn}V_{,nj}, \tag{60}$$

which represents, for given $V_{,ij}$, the linear ordinary differential equation for $V_{,\tilde{a}j}$. Equation (53) with (54) yields expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V_{,\tilde{a}\tilde{b}} = -V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}} \tag{61}$$

for the derivative of $V_{,\tilde{a}\tilde{b}}$ along the geodesic in terms of $V_{,\tilde{a}j}$. Equations (24) read

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial x^i}{\partial \tilde{x}^{\tilde{c}}} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}} + H^{,ik}\frac{\partial y_k}{\partial \tilde{x}^{\tilde{c}}}, \tag{62}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial x^i}{\partial \tilde{y}_{\tilde{c}}} = H_{,k}^{,i}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} + H^{,ik}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}}, \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y_i}{\partial \tilde{x}^{\tilde{c}}} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{x}^{\tilde{c}}}, \tag{64}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial y_i}{\partial \tilde{y}_{\tilde{c}}} = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}}. \tag{65}$$

4.4. Derivative of (27)

We express the derivative of Equation (27) along the geodesic using Equations (56), (57), (59), (63) and (65),

$$\begin{aligned}\Big[&-H_{,ij} - H_{,i}^{,m}V_{,mj} - V_{,im}H_{,j}^{,m} - V_{,im}H^{,mn}V_{,nj} - V^{-1}(H_{,i}V_{,j} + V_{,i}H_{,j}) - V^{-2}V_{,i}V_{,j}\Big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}\\ &+ (V_{,ij} + V^{-1}V_{,i}V_{,j})\left(H_{,k}^{,j}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} + H^{,jk}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}}\right) = -H_{,ik}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}} - H_{,i}^{,k}\frac{\partial y_k}{\partial \tilde{y}_{\tilde{c}}}.\end{aligned} \tag{66}$$

We consider the second-degree positive homogeneity of Hamiltonian function H,

$$\big[-H_{,ij}-H_{,i}^{,m}V_{,mj}-V_{,im}H_{,j}^{,m}-V_{,im}H^{,mn}V_{,nj}-V^{-1}(H_{,i}V_{,j}+V_{,i}H_{,j})-V^{-2}V_{,i}V_{,j}\big]\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}} \\ +(V_{,ij}H_{,k}^{,j}+2V^{-1}V_{,i}H_{,k})\tfrac{\partial x^k}{\partial\tilde{y}_{\tilde{c}}}+(V_{,ij}H^{,jk}+V^{-1}V_{,i}H^{,k})\tfrac{\partial y_k}{\partial\tilde{y}_{\tilde{c}}}=-H_{,ik}\tfrac{\partial x^k}{\partial\tilde{y}_{\tilde{c}}}-H_{,i}^{,k}\tfrac{\partial y_k}{\partial\tilde{y}_{\tilde{c}}}. \tag{67}$$

We collect the terms containing $\frac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}$ and the terms containing $\frac{\partial y_l}{\partial\tilde{y}_{\tilde{c}}}$,

$$\big[-H_{,i}^{,m}V_{,mj}-V_{,im}H^{,mn}V_{,nj}-V^{-1}H_{,i}V_{,j}+V^{-1}V_{,i}H_{,j}-V^{-2}V_{,i}V_{,j}\big]\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}} \\ +\big[V_{,im}H^{,mn}+V^{-1}V_{,i}H^{,n}+H_{,i}^{,n}\big]\tfrac{\partial y_n}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{68}$$

We insert (27) for $\frac{\partial y_l}{\partial\tilde{y}_{\tilde{c}}}$,

$$\big[-H_{,i}^{,m}V_{,mj}-V_{,im}H^{,ml}V_{,lj}-V^{-1}H_{,i}V_{,j}+V^{-1}V_{,i}H_{,j}-V^{-2}V_{,i}V_{,j} \\ (V_{,im}H^{,mn}+V^{-1}V_{,i}H^{,n}+H_{,i}^{,n})(V_{,nj}+V^{-1}V_{,n}V_{,j})\big]\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{69}$$

We consider the second-degree positive homogeneity of Hamiltonian function H,

$$\big[-H_{,i}^{,m}V_{,mj}-V_{,im}H^{,mn}V_{,nj}-V^{-1}H_{,i}V_{,j}+V^{-1}V_{,i}H_{,j}-V^{-2}V_{,i}V_{,j} \\ +V_{,im}H^{,mn}V_{,nj}+V^{-1}V_{,i}H^{,m}V_{,mj}+H_{,i}^{,m}V_{,mj} \\ +V^{-1}V_{,im}H^{,m}V_{,j}+V^{-2}V_{,i}V_{,j}+2V^{-1}H_{,i}V_{,j}\big]\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{70}$$

We now execute the summation in (70) and arrive at

$$\big[V^{-1}H_{,i}V_{,j}+V^{-1}V_{,i}H_{,j}+V^{-1}V_{,i}H^{,m}V_{,mj}+V^{-1}V_{,im}H^{,m}V_{,j}\big]\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{71}$$

This equation holds in consequence of Equation (49).

4.5. Derivative of (28)

We express the derivative of Equation (28) along the geodesic using Equations (56), (57), (58), (60) and (63),

$$\big(-V_{,\tilde{a}m}H_{,j}^{,m}-V_{,\tilde{a}m}H^{,mn}V_{,nj}-V^{-1}V_{,\tilde{a}}H_{,j}-V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}} \\ +\big(V_{,\tilde{a}j}+V^{-1}V_{,\tilde{a}}V_{,j}\big)\big(H_{,l}^{,j}\tfrac{\partial x^l}{\partial\tilde{y}_{\tilde{c}}}+H^{,jl}\tfrac{\partial y_l}{\partial\tilde{y}_{\tilde{c}}}\big)=0. \tag{72}$$

We consider the second-degree positive homogeneity of Hamiltonian function H,

$$\big(-V_{,\tilde{a}m}H_{,j}^{,m}-V_{,\tilde{a}m}H^{,mn}V_{,nj}-V^{-1}V_{,\tilde{a}}H_{,j}-V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}} \\ +\big(V_{,\tilde{a}j}H_{,l}^{,j}+2V^{-1}V_{,\tilde{a}}H_{,l}\big)\tfrac{\partial x^l}{\partial\tilde{y}_{\tilde{c}}}+\big(V_{,\tilde{a}j}H^{,jl}+V^{-1}V_{,\tilde{a}}H^{,l}\big)\tfrac{\partial y_l}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{73}$$

We collect the terms containing $\frac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}$ and the terms containing $\frac{\partial y_l}{\partial\tilde{y}_{\tilde{c}}}$,

$$\big(-V_{,\tilde{a}m}H^{,mn}V_{,nj}+V^{-1}V_{,\tilde{a}}H_{,j}-V^{-2}V_{,\tilde{a}}V_{,j}\big)\tfrac{\partial x^j}{\partial\tilde{y}_{\tilde{c}}}+\big(V_{,\tilde{a}m}H^{,mn}+V^{-1}V_{,\tilde{a}}H^{,n}\big)\tfrac{\partial y_n}{\partial\tilde{y}_{\tilde{c}}}=0. \tag{74}$$

We insert (27) for $\frac{\partial y_n}{\partial \tilde{y}_{\tilde{c}}}$,

$$\begin{aligned}&\big[-V_{,\tilde{a}m}H^{,mn}V_{,nj}+V^{-1}V_{,\tilde{a}}H_{,j}-V^{-2}V_{,\tilde{a}}V_{,j}\\&\quad+(V_{,\tilde{a}m}H^{,mn}+V^{-1}V_{,\tilde{a}}H^{,n})(V_{,nj}+V^{-1}V_{,n}V_{,j})\big]\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}=0.\end{aligned} \tag{75}$$

We consider the second-degree positive homogeneity of Hamiltonian function H,

$$\begin{aligned}&\big(-V_{,\tilde{a}m}H^{,mn}V_{,nj}+V^{-1}V_{,\tilde{a}}H_{,j}-V^{-2}V_{,\tilde{a}}V_{,j}\\&\quad+V_{,\tilde{a}m}H^{,mn}V_{,nj}+V^{-1}V_{,\tilde{a}}H^{,n}V_{,nj}+V^{-1}V_{,\tilde{a}m}H^{,m}V_{,j}+V^{-2}V_{,\tilde{a}}V_{,j}\big)\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}=0.\end{aligned} \tag{76}$$

We now execute the summation in (76) and arrive at

$$\big(V^{-1}V_{,\tilde{a}}H_{,j}+V^{-1}V_{,\tilde{a}}H^{,n}V_{,nj}+V^{-1}V_{,\tilde{a}m}H^{,m}V_{,j}\big)\frac{\partial x^j}{\partial \tilde{y}_{\tilde{c}}}=0. \tag{77}$$

This equation holds in consequence of Equations (49) and (50).

4.6. Derivative of (29)

We express the derivative of Equation (29) along the geodesic using Equations (56), (58), (61), (62) and (63),

$$\begin{aligned}&\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(-V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}}-V^{-2}V_{,\tilde{a}}V_{,\tilde{b}}\big)\\&\quad+\Big(H^{,i}_{,m}\frac{\partial x^m}{\partial \tilde{y}_{\tilde{a}}}+H^{,im}\frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\Big)\big(V_{,\tilde{a}\tilde{b}}+V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big)=H^{,i}_{,m}\frac{\partial x^m}{\partial \tilde{x}^{\tilde{b}}}+H^{,im}\frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}}.\end{aligned} \tag{78}$$

Considering Equation (29), the terms containing $H^{,i}_{,m}$ cancel out,

$$\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(-V_{,\tilde{a}m}H^{,mn}V_{,n\tilde{b}}-V^{-2}V_{,\tilde{a}}V_{,\tilde{b}}\big)+H^{,im}\frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}\tilde{b}}+V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big)=H^{,im}\frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}}. \tag{79}$$

Considering (50) and the second-degree positive homogeneity of H, we may express Equation (79) as

$$\begin{aligned}&-\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}m}+V^{-1}V_{,\tilde{a}}V_{,m}\big)H^{,mn}\big(V_{,n\tilde{b}}+V^{-1}V_{,n}V_{,\tilde{b}}\big)\\&\quad+H^{,im}\frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}\tilde{b}}+V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big)=H^{,im}\frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}}.\end{aligned} \tag{80}$$

We multiply Equation (80) from the right-hand side by matrix $\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}$,

$$\begin{aligned}&-\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}m}+V^{-1}V_{,\tilde{a}}V_{,m}\big)H^{,mn}\big(V_{,n\tilde{b}}+V^{-1}V_{,n}V_{,\tilde{b}}\big)\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}\\&\quad+H^{,im}\frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\big(V_{,\tilde{a}\tilde{b}}+V^{-1}V_{,\tilde{a}}V_{,\tilde{b}}\big)\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}=H^{,im}\frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}.\end{aligned} \tag{81}$$

Equation (28) implies

$$\frac{\partial x^i}{\partial \tilde{y}_{\tilde{a}}}\Big(V_{,\tilde{a}j}+V^{-1}V_{,\tilde{a}}V_{,j}\Big)=-\delta^i_j. \tag{82}$$

We insert (29) and (82) into Equation (81) and obtain

$$-H^{,ik}+H^{,im}\frac{\partial y_m}{\partial \tilde{y}_{\tilde{a}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{a}}}=H^{,im}\frac{\partial y_m}{\partial \tilde{x}^{\tilde{b}}}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{b}}}. \tag{83}$$

Equation (83) holds thanks to the consequence

$$\frac{\partial y_i}{\partial \tilde{y}_{\tilde{c}}}\frac{\partial x^k}{\partial \tilde{x}^{\tilde{c}}}-\frac{\partial y_i}{\partial \tilde{x}^{\tilde{c}}}\frac{\partial x^k}{\partial \tilde{y}_{\tilde{c}}}=\delta^k_i \tag{84}$$

of the symplectic property of matrix (22).

5. Conclusions

The propagator matrix (22) of geodesic deviation contains all the linearly independent solutions of the equations of geodesic deviation. It can be calculated using the Hamiltonian equations (14) and (15) of geodesic deviation which are considerably simpler than the Riemannian or Finslerian equations (19) and (20) of geodesic deviation.

In the high-frequency approximations of wave propagation, the propagator matrix (22) of geodesic deviation may be used to calculate the wave amplitude for any initial conditions. The amplitude of the Green function of the corresponding wave field is proportional to expression $V^{\frac{1}{2}}\big|\det\big(\frac{\partial x^i}{\partial \tilde{y}_j}\big)\big|^{-\frac{1}{2}} = \big|V\det\big(\frac{\partial V}{\partial \tilde{x}^i \partial x^j} + V^{-1}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial x^j}\big)\big|^{\frac{1}{2}}$.

The main result of this paper, relations (27)–(29) between the propagator matrix (22) of geodesic deviation and the second-order spatial derivatives of the characteristic function are applicable to the Finsler geometry, to the Riemann geometry, and to their various applications such as general relativity or the high-frequency approximations of wave propagation, see [14].

The equations, derived in this paper for the Hamiltonian function positively homogeneous of the second degree with respect to the spatial gradient of the characteristic function, were generalized to an arbitrary Hamiltonian function by Klimeš, see [7].

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