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Perturbation Hamiltonians in heterogeneous anisotropic weakly dissipative media

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SUMMARY

Perturbation Hamiltonians simplify considerably the solution of various traveltime perturbation problems of seismic body waves propagating in heterogeneous, isotropic or anisotropic, weakly dissipative media. In addition to the phase-space coordinates, representing the Cartesian coordinates and Cartesian components of the slowness vector, the perturbation Hamiltonians also depend on one or more dimensionless perturbation parameters, which can be chosen in different ways. General traveltime perturbation procedures, based on the perturbation Hamiltonians, were proposed by Klimeš. The evaluation of the perturbation expansions for the traveltime and for its derivatives does not require the computation of perturbed rays in these procedures. It consists in dynamic ray tracing along the reference ray, in the determination of the partial derivatives of the perturbation Hamiltonian with respect to the perturbation parameters, and in additional quadratures along a reference ray, computed in the reference medium. Different forms of the perturbation Hamiltonian yield different perturbation expansions with different accuracy and numerical efficiency of computations. In this paper, the special case of the perturbation Hamiltonian, herein referred to as the linear perturbation Hamiltonian, is used to compute the second-order perturbation expansion of the complex-valued traveltime and the first-order perturbation expansion of the complex-valued traveltime gradient. The advantage of the linear perturbation Hamiltonian is that it is linear with respect to perturbation parameters. Heterogeneous, perfectly elastic, isotropic or anisotropic, reference media are considered, in which reference rays can be calculated without problems by well-known methods. The derived perturbation expansions are valid very generally. For the isotropic reference model, the perturbed medium may be isotropic perfectly elastic, isotropic weakly dissipative, weakly anisotropic perfectly elastic or weakly anisotropic weakly dissipative. For the anisotropic reference model, the perturbed medium may be anisotropic perfectly elastic, or anisotropic weakly dissipative. Expressions for the attenuation vector, the dissipation filter, the global absorption factor, the direction-dependent scalar local quality factor and the attenuation coefficient are also derived for seismic body waves propagating in heterogeneous anisotropic weakly dissipative media.

Key words: Seismic anisotropy; Seismic attenuation; Theoretical seismology; Wave propagation.

1 INTRODUCTION

In forward modelling and inversion of traveltime fields of seismic body waves propagating in heterogeneous, isotropic or anisotropic media, perturbation methods play an important role. The perturbation methods are based on the assumption that the traveltime in a reference model is known or may be simply computed, and that the perturbed model differs only slightly from the reference model. The traveltime in the perturbed model we wish to compute is then expanded in terms of a power series in the perturbation parameters. If we retain only the terms up to the nth power in the series, we speak of the nth-order perturbation expansion, and call the kth term in the perturbation expansion the kth-order perturbation. Similar perturbation expansions may also be found for traveltime derivatives, for example, for the traveltime gradient.

Commonly, the perturbation parameters represent the differences between elastic (or viscoelastic) moduli in the perturbed and reference medium. They may, however, be taken also in different ways, as demonstrated in this paper.

Considerable attention in the literature has been devoted to the first-order and second-order perturbations of the traveltime, and to the first-order perturbation of the traveltime gradient. The great interest in the first-order perturbation of the traveltime is not surprising. The first-order traveltime expansion often yields sufficiently accurate results. At the same time, it is relatively simple in comparison with higher-order perturbations and/or with the first-order perturbation of traveltime derivatives.

Several approaches have been used in the traveltime perturbation methods. One approach is based on Fermat's principle, see Aki & Richards (1980) and Nolet (1987) for isotropic media and Chapman & Pratt (1992) for anisotropic media. The approach has been used mostly to compute first-order traveltime perturbations. Another approach is based on linearizing the eikonal equation and/or on the first-order perturbation of the Hamiltonian. For anisotropic media, see Červený (1982), Červený & Jech (1982), Hanyga (1982), Farra & Madariaga (1987), Nowack & Lutter (1988), Farra (1989), Farra *et al.* (1989), Jech & Pšenčík (1989), Nowack & Pšenčík (1991) and Farra & Le Bégat (1995). This approach is again suitable for determining first-order traveltime perturbation. Supplemented by dynamic ray tracing along the reference ray, however, the method can also be used in computing paraxial rays, and, consequently, in computing the second-order perturbation of traveltimes (Farra & Le Bégat 1995; Farra 1999; Červený 2001), and of first-order perturbation of the traveltime gradient (Červený 2002). Another approach is based on the first-order perturbation of the Hamiltonian, and on the use of the first-order Hamiltonian for ray tracing and dynamic ray tracing. The traveltime along such first-order rays is of the first-order, and can be corrected by quadratures to the second order (Pšenčík & Farra 2005, 2007; Farra & Pšenčík 2008). Yet another approach is based on the Lagrangian formulation of the ray equations, see Snieder & Sambridge (1992), Snieder & Aldridge (1995), etc.

Finally, the approach, proposed by Klimeš (2002), is based on the perturbation Hamiltonians. The perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\kappa)$ is a function of the phase coordinates, consisting of the Cartesian coordinates x_a and Cartesian components of the slowness vector p_b, and of one or several dimensionless perturbation parameters f_κ, $\kappa = 1, 2 \ldots$. The procedure of determining traveltime perturbations is based on the quadratures along a reference ray. It does not require the computation of perturbed rays. It is quite universal and general, and may be recursively used to calculate the perturbations of any order of the traveltime and of the perturbations of any order of partial derivatives of the traveltime of any order. In the literature, mostly the perturbation Hamiltonians, which are homogeneous functions of degrees $N = 2$ or -1 in p_b, have been used, although arbitrary degree N or even non-homogeneous Hamiltonians may also be applied. Klimeš. & Bulant (2004, 2006) and Bulant & Klimeš (2008) used the method of the perturbation Hamiltonian successfully in studying the coupling ray theory of S waves in anisotropic media.

In all above listed papers, the real-valued traveltime was considered. The perturbation Hamiltonian, however, can be used even to compute complex-valued traveltimes, corresponding to weakly dissipative media. The reference medium is perfectly elastic and thus real-valued, and the reference rays are also real-valued. Consequently, standard well-known methods may be used to compute the reference rays, and a weakly dissipative medium is considered as a perturbation of the perfectly elastic medium. The method of the perturbation Hamiltonian was used to investigate the properties of attenuation vector of seismic body waves in heterogeneous anisotropic dissipative media by Červený, Klimeš & Pšenčík (2008), where also the term 'perturbation Hamiltonian' was first used. The homogeneous perturbation Hamiltonian of the degree $N = 2$ in p_b was considered.

The perturbation Hamiltonian may be constructed in different forms. Different forms of the perturbation Hamiltonian yield different expressions for the perturbation of traveltime of a given order with different accuracy and numerical efficiency of computations. In this paper, we use a special form of the perturbation Hamiltonian, called here the linear perturbation Hamiltonian. It is linear with respect to perturbation parameters, but need not be linear with respect to viscoelastic moduli. Consequently, it promises high accuracy of results. Note that the linear perturbation Hamiltonian used here is different from the perturbation Hamiltonian used in Červený *et al.* (2008).

The main purpose of this paper is to use the linear perturbation Hamiltonian to derive and discuss the expressions for the first-order and second-order perturbations of complex-valued traveltime and for the first-order perturbation of complex-valued traveltime gradient. We consider general homogeneous Hamiltonians of the Nth degree in p_a, where N may be an arbitrary integer. We also include the discussion of the attenuation vector, of the global absorption factor, of the local quality factor and of the attenuation coefficient of body waves propagating in heterogeneous anisotropic weakly dissipative media.

The content of this paper is as follows. In Section 2, perturbation Hamiltonians are introduced, and the linear perturbation Hamiltonian is defined. The definition is very general; it includes heterogeneous, isotropic or anisotropic, perfectly elastic, reference media, and heterogeneous, isotropic or anisotropic, perfectly elastic or weakly dissipative perturbed media. In Section 3, expressions for the perturbation expansions for the complex-valued traveltime and for the complex-valued traveltime gradient are summarized. In Section 4, all perturbation expansions are expressed in terms of the linear perturbation Hamiltonian. In Section 5, special cases are discussed. In Section 5.1, we discuss the case in which the perturbation of elastic moduli is purely imaginary. Consequently, the real parts of the density-normalized viscoelastic moduli in the perturbed media equal the elastic moduli in the reference medium. The imaginary parts are zero in the reference medium. In Section 5.2, a heterogeneous, perfectly elastic, isotropic reference model is considered. The perturbed medium may then be isotropic perfectly elastic, isotropic weakly dissipative, weakly anisotropic perfectly elastic or weakly anisotropic weakly dissipative. As isotropic media are very important in practical applications, we present the complete explicit equations for the case of the isotropic and perfectly elastic reference medium, and for the isotropic, perfectly elastic or weakly dissipative perturbed medium.

We denote Cartesian coordinates x_i and time t. We consider high-frequency time-harmonic seismic body waves, with the exponential factor $\exp(-\mathrm{i}\omega\, t)$, where ω is fixed, real-valued circular frequency. The lower-case Roman indices take the values 1, 2, 3, the upper-case indices 1, 2. The Greek indices take any positive integer value. The Einstein summation convention over repeating Roman indices is used.

2 HAMILTONIANS

2.1 Reference Hamiltonians

Traveltime $\tau(x_i)$ of any high-frequency seismic body wave propagating in a smoothly heterogeneous, isotropic or anisotropic medium is fully described by the eikonal equation and by the relevant initial conditions. The eikonal equation can be derived from the equation of motion by high-frequency asymptotic methods. It is usual to express the eikonal equation in Hamiltonian form

$$\mathcal{H}(x_a, p_b) = c, \tag{1}$$

where $\mathcal{H}$ is the Hamiltonian, x_a and p_b are phase-space coordinates, consisting of the Cartesian coordinates x_a and the Cartesian components of the slowness vector, $p_b = \partial\tau/\partial x_b$, and c is a real-valued constant. For perfectly elastic media, the Hamiltonian is defined in a real-valued 6-D phase-space x_a, p_b $(a, b = 1, 2, 3)$, where x_a and p_b are real valued. For viscoelastic media, complex-valued x_a and p_b should be considered, and the 6-D phase space is complex-valued (Thomson 1997).

We use the following notations for the first and second partial derivatives of Hamiltonian $\mathcal{H}(x_a, p_b)$

$$\mathcal{H}^{,i} = \partial\mathcal{H}/\partial p_i, \qquad \mathcal{H}_{,i} = \partial\mathcal{H}/\partial x_i,$$
$$\mathcal{H}^{,ij} = \partial^2\mathcal{H}/\partial p_i\partial p_j, \quad \mathcal{H}^{,i}_{,j} = \partial^2\mathcal{H}/\partial p_i\partial x_j, \quad \mathcal{H}_{,ij} = \partial^2\mathcal{H}/\partial x_i\partial x_j. \tag{2}$$

Here the index i in the superscript denotes the partial derivative of the Hamiltonian with respect to p_i, and the index i in the subscript denotes the partial derivative of the Hamiltonian with respect to x_i.

Eq. (1) is a non-linear partial differential equation of the first order. We consider here only Hamiltonians, which are homogeneous functions of the Nth degree in p_b. Actually, it would be possible to consider arbitrary Hamiltonians, but we shall limit ourselves to the simplest case of homogeneous Hamiltonians, to make the treatment most transparent. For anisotropic heterogeneous media, we can use the Hamiltonian of the Nth degree in p_b in the form

$$\mathcal{H}(x_a, p_b) = \frac{1}{N}[G_m(x_a, p_b)]^{N/2}. \tag{3}$$

Here $G_m(x_a, p_b)$ is one of the three eigenvalues of the 3×3 generalized Christoffel matrix Γ_{ik}

$$\Gamma_{ik}(x_a, p_b) = a_{ijkl}(x_a)p_j p_l. \tag{4}$$

For simplicity, we call it the Christoffel matrix in what follows. Here $a_{ijkl}(x_n)$ are spatially variable density-normalized elastic or viscoelastic moduli, satisfying the symmetry relations

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}. \tag{5}$$

The viscoelastic moduli are frequency dependent. We, however, consider an arbitrarily selected, but fixed ω. Subscript m $(m = 1, 2, 3)$ in $G_m(x_a, p_b)$ indicates the mode of the wave under consideration (P, S1 and S2). We also denote by $\mathbf{g}^{(m)}(x_a, p_b)$ the relevant eigenvector of the Christoffel matrix (4), corresponding to the eigenvalue $G_m(x_a, p_b)$. For a selected wave mode (P, S1 or S2), eigenvector $\mathbf{g}^{(m)}$ is a function of $\mathbf{x}$ and $\mathbf{p}$. The Cartesian components of $\mathbf{g}^{(m)}$ satisfy the system of three linear equations

$$\Gamma_{ik}g_k^{(m)} = g_i^{(m)}, \tag{6}$$

and the normalization condition,

$$\mathbf{g}^{(m)}\mathbf{g}^{(m)} = 1 \tag{7}$$

(no summation over m). Consequently, the eigenvector $\mathbf{g}^{(m)}(x_a, p_b)$ is a real-valued or complex-valued unit vector. We can then express eigenvalue $G_m(x_a, p_b)$ in the following form

$$G_m(x_a, p_b) = \Gamma_{ik}(x_a, p_b)g_i^{(m)}g_k^{(m)} = a_{ijkl}(x_n)p_j p_l g_i^{(m)}g_k^{(m)} \tag{8}$$

(no summation over m). It follows from the equation of motion that eigenvalue $G_m(x_a, p_b)$ of any selected wave mode satisfies, in the phase space, the relation (Červený 2001, eq. 3.6.2)

$$G_m(x_a, p_b) = a_{ijkl}(x_n)p_j p_l g_i^{(m)}g_k^{(m)} = 1 \tag{9}$$

(no summation over m). For formulae of the partial derivatives of $G_m(x_a, p_b)$ and $\mathcal{H}(x_a, p_b)$ in anisotropic media see Červený (2001, section 4.14.1) and Klimeš (2006b).

Using eq. (3) for the Hamiltonian, we can compute any ray. The ray tracing equations read

$$\mathrm{d}x_i/\mathrm{d}\gamma_3 = \mathcal{H}^{,i}, \quad \mathrm{d}p_i/\mathrm{d}\gamma_3 = -\mathcal{H}_{,i}. \tag{10}$$

$\mathcal{H}^{,i}$ and $\mathcal{H}_{,i}$ are partial derivatives of the Hamiltonian with respect to p_i and x_i, see (2), and γ_3 is the variable along the ray.

In perfectly elastic media, the variable γ_3 along the ray is real-valued, and the computation of rays is a well-understood problem, both for isotropic and anisotropic media (Červený 2001). In viscoelastic media, position vector $\mathbf{x}$, slowness vector $\mathbf{p}$ and variable γ_3 would be complex-valued, and the ray tracing would be considerably more complicated than in perfectly elastic media (Thomson 1997).

In this paper, however, we consider only weakly dissipative media and systematically use a perfectly elastic reference medium, with a real-valued reference Hamiltonian. We use the perturbation approach, and consider the weakly dissipative medium as a small perturbation of the perfectly elastic medium. We then do not need to use the Hamiltonian (1) and ray tracing system (10) in a complex-valued 6-D phase space with complex-valued x_i, p_i and γ_3. It is sufficient to perform standard ray tracing in a reference perfectly elastic medium, which yields real-valued rays.

We now use the Hamiltonian (3) and the relation (9) in the reference, perfectly elastic media. It can be then proved that all expressions in ray tracing system (10) are real-valued and independent of N. For example, $\mathcal{H}^{,i}$ and $\mathcal{H}_{,i}$ are given by N-independent expressions, see (3)

$$\mathcal{H}^{,i} = \frac{1}{2}\partial G/\partial p_i, \quad \mathcal{H}_{,i} = \frac{1}{2}\partial G/\partial x_i. \tag{11}$$

We introduce the standard notation

$$\mathcal{U}_i = \mathcal{H}^{,i} = \frac{1}{2}\partial G/\partial p_i, \tag{12}$$

and call the vector $\boldsymbol{\mathcal{U}}$, with Cartesian components $\mathcal{U}_i$, the ray-velocity vector. It is tangent to the ray, and its magnitude is called the ray velocity $\mathcal{U}$. The real-valued, N-independent variable γ_3 represents the traveltime along the ray.

Along real-valued rays, however, we need to perform dynamic ray tracing and perform some additional quadratures. To perform dynamic ray tracing, we specify each real-valued ray by ray parameters γ_1, γ_2, which may represent the take-off angles at a point source, or Gaussian coordinates along an initial surface. We introduce the 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$, with components Q_{ia} and P_{ia}, by the relations

$$Q_{ia} = \partial x_i/\partial\gamma_a, \quad P_{ia} = \partial p_i/\partial\gamma_a. \tag{13}$$

These matrices play a very important role in the perturbation method described in this paper. They can be computed by dynamic ray tracing along the known ray (or together with ray tracing equations)

$$\mathrm{d}Q_{ia}/\mathrm{d}\gamma_3 = \mathcal{H}^{,i}_{,j}Q_{ja} + \mathcal{H}^{,ij}P_{ja}, \quad \mathrm{d}P_{ia}/\mathrm{d}\gamma_3 = -\mathcal{H}_{,ij}Q_{ja} - \mathcal{H}^{,j}_{,i}P_{ja}. \tag{14}$$

Here $\mathcal{H}^{,i}_{,j}$, $\mathcal{H}^{,ij}$ and $\mathcal{H}_{,ij}$ denote the second partial derivatives of the Hamiltonian, given by (2). They do depend on N. Consequently, the dynamic ray tracing and its results also depend on N.

It is useful to express the 3×3 matrix $\mathbf{Q}$ in terms of three real-valued contravariant basis vectors $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ connected with the reference ray Ω

$$\mathbf{Q} \equiv [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \equiv \boldsymbol{\mathcal{U}}]. \tag{15}$$

Here $\mathcal{U}$ is the real-valued ray-velocity vector. Vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are tangent to the wave front in the reference perfectly elastic medium, $\mathbf{Q}_3$ is tangent to the ray. The ith Cartesian component of vector $\mathbf{Q}_k$ is Q_{ik}, and the Cartesian components of $\mathcal{U}$ are Q_{i3}.

The initial point of the ray may represent a point source, or may be situated on an initial surface, on an initial line, or in a smooth medium. The relevant initial conditions for ray tracing system (10) and for dynamic ray tracing system (14) may be found in Červený & Moser (2009). For more details on ray tracing and dynamic ray tracing in perfectly elastic media, see Červený (2001) and Červený *et al.* (2007).

In the following, we consider an arbitrary perfectly elastic medium, specified by real-valued, smoothly varying, density-normalized elastic moduli a_{ijkl}, and call it the reference medium, and the relevant Hamiltonian the reference Hamiltonian. We further consider an arbitrary, real-valued ray in the reference medium and call it the reference ray Ω. We assume that the real-valued slowness vector $\mathbf{p}$, eigenvector $\mathbf{g}$, ray-velocity vector $\mathcal{U}$, the 3×3 real-valued matrices $\mathbf{Q}$ and $\mathbf{P}$, and the partial derivatives $\mathcal{H}^{,i}, \mathcal{H}_{,i}, \mathcal{H}^{,ij}, \mathcal{H}^{,i}_{,j}, \mathcal{H}^{,j}_{,i}, \mathcal{H}_{,ij}$ of the reference Hamiltonian, are known along ray Ω, or can be simply calculated along it.

2.2 Perturbation Hamiltonians

Now we consider another medium, which differs only slightly from the reference one, and call it the perturbed medium. We describe the density-normalized moduli in the perturbed medium by the expression

$$a_{ijkl}(x_n) + b_{ijkl}(x_n), \tag{16}$$

where $b_{ijkl}(x_n)$ are smoothly varying perturbations of the density-normalized elastic moduli $a_{ijkl}(x_n)$. Perturbations $b_{ijkl}(x_n)$ should satisfy symmetry relations analogous to (5). They are real-valued when both the reference and perturbed medium are perfectly elastic. If $b_{ijkl}(x_n)$ are complex-valued, the perturbed medium is dissipative.

For complex-valued b_{ijkl}, we use notation

$$b_{ijkl} = b^R_{ijkl} - \mathrm{i} b^I_{ijkl}. \tag{17}$$

The minus sign in (17) is related to the minus sign in the exponential factor $\exp(-\mathrm{i}\omega t)$ of the wave under consideration. We consider only such 'physical' media, for which matrices a_{ijkl} and $a_{ijkl} + b^R_{ijkl}$ in the Voigt notation are positive definite (Fedorov 1968; Musgrave 1970; Helbig 1994), and b^I_{ijkl} are positive-definite or zero (Červený & Pšenčík 2005). We call b^I_{ijkl} the dissipation moduli. The case of $b^I_{ijkl} = 0$ corresponds to the perfectly elastic perturbed model.

The perturbation Hamiltonian, introduced to the seismic traveltime perturbation theory by Klimeš (2002), reads

$$\mathcal{H} = \mathcal{H}(x_a, p_b, f_\kappa). \tag{18}$$

For fixed f_κ, the perturbation Hamiltonian is defined in a 6-D phase space x_a, p_b, where x_a are real-valued Cartesian coordinates and p_b real-valued components of the slowness vector, both along the reference ray in the reference medium. The perturbation Hamiltonian contains, in addition to x_a and p_b, one or several perturbation parameters f_κ ($\kappa = 1, 2, \ldots$). For $f_\kappa = 0$ ($\kappa = 1, 2, \ldots$), the perturbation Hamiltonian reduces to the reference Hamiltonian.

The perturbation Hamiltonian can be used to compute the perturbation of any order of traveltime $\tau(x_i)$ and of the partial derivatives of the traveltime of any order. All the quantities in the perturbed medium are obtained just by quadratures along the reference ray computed in the reference medium. No computation of rays in the perturbed medium is required. This is very important, particularly if the perturbed medium is dissipative (b_{ijkl} complex-valued). The complex-valued ray tracing would complicate the algorithm considerably. The perturbation method is, of course, only approximate, and valid for weakly dissipative media only. The dissipation in the Earth's interior is, however, usually weak, and thus the perturbation method can be used quite broadly.

The perturbation Hamiltonian can be constructed in various ways. We shall consider here homogeneous perturbation Hamiltonians specified by one perturbation parameter f_α only. Two simple forms of such perturbation Hamiltonians are as follows.

The first suitable form of the perturbation Hamiltonian of degree $N = 2$, valid for smoothly variable anisotropic, perfectly elastic or viscoelastic media, is given by the relation

$$\mathcal{H}(x_a, p_b, f_\alpha) = \frac{1}{2}(a_{ijkl} + f_\alpha b_{ijkl}) p_j p_l \bar{g}_i \bar{g}_k. \tag{19}$$

Here $\bar{\mathbf{g}}$ is the eigenvector of the perturbed Christoffel matrix $(a_{ijkl} + f_\alpha b_{ijkl}) p_j p_l$. We can see that it is a function of perturbation parameter f_α. Thus $\bar{\mathbf{g}}$ has to be expanded in terms of f_α in (19). The components of slowness vector $\mathbf{p}$, however, do not depend on f_α; they correspond to the reference medium a_{ijkl}. Červený *et al.* (2008) used this homogeneous perturbation Hamiltonian, with imaginary valued b_{ijkl}, to study the attenuation vector in heterogeneous anisotropic weakly dissipative media. It yielded compact perturbation expressions for the attenuation vector and for the complex-valued ray-velocity vector along the reference ray. Note that perturbation Hamiltonian (19) is a non-linear function of perturbation parameter f_α because $\bar{\mathbf{g}} = \bar{\mathbf{g}}(f_\alpha)$.

The second form of the perturbation Hamiltonian of degree N, valid also for smoothly variable anisotropic perfectly elastic or viscoelastic media, is given by the relation

$$\mathcal{H}(x_a, p_b, f_\alpha) = \mathcal{H}^0(x_a, p_b) + f_\alpha \Delta\mathcal{H}(x_a, p_b), \tag{20}$$

where

$$\Delta\mathcal{H}(x_a, p_b) = \tilde{\mathcal{H}}(x_a, p_b) - \mathcal{H}^0(x_a, p_b) \tag{21}$$

with

$$\mathcal{H}^0(x_a, p_b) = \frac{1}{N}[a_{ijkl} p_j p_l g_i g_k]^{N/2}, \tag{22}$$

$$\tilde{\mathcal{H}}(x_a, p_b) = \frac{1}{N}[(a_{ijkl} + b_{ijkl}) p_j p_l \tilde{g}_i \tilde{g}_k]^{N/2}. \tag{23}$$

Here $\mathcal{H}^0(x_a, p_b)$ is the reference Hamiltonian of degree N, the slowness vector $\mathbf{p}$ corresponds again to the reference medium, and $\tilde{\mathbf{g}}$ is one of the eigenvectors of the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}(x_a, p_b)$, given by the relation

$$\tilde{\Gamma}_{ik}(x_a, p_b) = (a_{ijkl} + b_{ijkl}) p_j p_l. \tag{24}$$

For real-valued b_{ijkl}, the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ is real valued, and the eigenvectors $\tilde{\mathbf{g}}$ are also real-valued. For complex-valued b_{ijkl}, the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ is complex-valued, and eigenvectors $\tilde{\mathbf{g}}$ are also complex-valued. We call (20) the linear perturbation Hamiltonian in this paper. The term 'linear' emphasizes that the perturbation Hamiltonian (20) is a linear function of perturbation parameter f_α in contrast to perturbation Hamiltonian (19).

Perturbation Hamiltonians (19) and (20) would coincide for $N = 2$, $f_\alpha = 1$ and $\tilde{\mathbf{g}} = \mathbf{g}$ in (22) and (23). The results corresponding to the linear perturbation Hamiltonians for different specifications of N, however, may be more general and more accurate than the results corresponding to perturbation Hamiltonian (19). The main reason is that the linear perturbation Hamiltonian (20) is linear in perturbation parameter f_α. Another reason is the use of the exact eigenvector $\tilde{\mathbf{g}}$ in (20). It is more accurate than $\bar{\mathbf{g}}$ used in (19).

3 THE PERTURBATION EXPANSION IN TERMS OF THE PERTURBATION HAMILTONIAN

Traveltime τ in the perturbed medium is a function of spatial coordinates x_i and of perturbation parameter f_α

$$\tau = \tau(x_m, f_\alpha). \tag{25}$$

A similar expression holds for the partial derivatives of τ of any order. For example, for the traveltime gradient $\tau_{,i} = \partial\tau/\partial x_i$

$$\tau_{,i} = \tau_{,i}(x_m, f_\alpha). \tag{26}$$

Expressions (25) and (26) can be expanded in terms of f_α. We are interested only in the second-order perturbation expansion of traveltime $\tau(x_m, f_\alpha)$, and in the first-order perturbation expansion of traveltime gradient $\tau_{,i}(x_m, f_\alpha)$. These perturbation expansions are the most important in seismic applications. They read

$$\tau(x_m, f_\alpha) \approx \tau(x_m) + f_\alpha\tau_{,\alpha}(x_m) + \frac{1}{2}(f_\alpha)^2\tau_{,\alpha\alpha}(x_m), \tag{27}$$

$$\tau_{,i}(x_m, f_\alpha) \approx \tau_{,i}(x_m) + f_\alpha\tau_{,i\alpha}(x_m). \tag{28}$$

Here $\tau_{,\alpha}(x_m)$, $\tau_{,\alpha\alpha}(x_m)$ and $\tau_{,i\alpha}(x_m)$ denote the perturbation derivatives at $f_\alpha = 0$, given by the expressions

$$\begin{aligned}
\tau_{,\alpha}(x_m) &= [\partial\tau(x_m, f_\alpha)/\partial f_\alpha]_{f_\alpha=0},\\
\tau_{,i\alpha}(x_m) &= [\partial^2\tau(x_m, f_\alpha)/\partial x_i\partial f_\alpha]_{f_\alpha=0},\\
\tau_{,\alpha\alpha}(x_m) &= [\partial^2\tau(x_m, f_\alpha)/\partial f_\alpha^2]_{f_\alpha=0}.
\end{aligned} \tag{29}$$

In the final expressions for $\tau(x_m, f_\alpha)$ and $\tau_{,i}(x_m, f_\alpha)$, given by (27) and (28), the perturbation parameter f_α equal to 1 can be used. In such a case, the perturbation expansions correspond to the medium specified by the density-normalized moduli (16).

The basic problem in the application of the perturbation expansions (27), (28) consists in the determination of the perturbation derivatives (29). The procedures of computing the perturbation derivatives of any order of the traveltime or of a partial derivative of the traveltime of any order were proposed by Klimeš (2002). The perturbation derivatives can be determined by quadratures along the reference ray, using perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ and its derivatives, and the 3×3 matrices $\mathbf{Q}$, $\mathbf{P}$, known from dynamic ray tracing, see (13), (14).

The algorithm proposed by Klimeš (2002) is very general and can be used to compute the perturbation expansions of traveltime of any order. The algorithm is recursive. When computing the higher-order spatial and perturbation derivatives, we have to know the partial derivatives of lower orders. Once perturbation derivatives (29) are known, it is easy to determine perturbation expansions (27) and (28), valid also in some f_α-vicinity of the real-valued reference ray. For the complex-valued perturbations b_{ijkl} of the density-normalized elastic moduli, the f_α-vicinity is complex-valued.

As we need to know the perturbation derivatives only along the reference ray, we can parametrize them by γ_3, which uniquely specifies any point on the ray. The real-valued slowness vector $\mathbf{p}$ can also be parametrized by γ_3. We also assume that $\tau_{,\alpha}(\gamma_3^0) = 0$ and $\tau_{,\alpha\alpha}(\gamma_3^0) = 0$, expressing the fact that the initial traveltime is not perturbed. Note, however, that $\tau_{,i\alpha}(\gamma_3^0)$ is generally non-zero. Klimeš' (2002) algorithm for evaluating perturbation derivatives $\tau_{,\alpha}(\gamma_3)$, $\tau_{,i\alpha}(\gamma_3)$ and $\tau_{,\alpha\alpha}(\gamma_3)$ is then simple. For a detailed derivation and explanations see Červený *et al.* (2008). The sought expressions are

$$\tau_{,\alpha}(\gamma_3) = T_\alpha(\gamma_3), \quad \tau_{,\alpha\alpha}(\gamma_3) = T_{\alpha\alpha}(\gamma_3), \tag{30}$$

$$\tau_{,i\alpha}(\gamma_3) = T_{k\alpha}(\gamma_3)Q^{-1}_{ki}(\gamma_3). \tag{31}$$

$T_\alpha(\gamma_3)$, $T_{k\alpha}(\gamma_3)$ and $T_{\alpha\alpha}(\gamma_3)$ are given by the relations

$$\begin{aligned}
T_\alpha(\gamma_3) &= -\int_{\gamma_3^0}^{\gamma_3} \mathcal{H}_{,\alpha}\,\mathrm{d}\gamma_3',\\
T_{K\alpha}(\gamma_3) &= T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3} \left[\mathcal{H}_{,i\alpha}Q_{iK} + \mathcal{H}^{;i}_{,\alpha}P_{iK}\right]\mathrm{d}\gamma_3',\\
T_{3\alpha}(\gamma_3) &= -\mathcal{H}_{,\alpha}(\gamma_3),\\
T_{\alpha\alpha}(\gamma_3) &= -\int_{\gamma_3^0}^{\gamma_3} \left[\mathcal{H}_{,\alpha\alpha} + 2\mathcal{H}^{;i}_{,\alpha}\tau_{,i\alpha} + \mathcal{H}^{;ij}\tau_{,i\alpha}\tau_{,j\alpha}\right]\mathrm{d}\gamma_3'.
\end{aligned} \tag{32}$$

Note that $T_\alpha(\gamma_3^0) = T_{\alpha\alpha}(\gamma_3^0) = 0$. Quantities $\mathcal{H}_{,\alpha}, \mathcal{H}_{,i\alpha}, \mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,\alpha\alpha}$ denote the perturbation derivatives of the perturbation Hamiltonian

$$\begin{aligned}
\mathcal{H}_{,\alpha} &= \partial\mathcal{H}/\partial f_\alpha, & \mathcal{H}_{,\alpha\alpha} &= \partial^2\mathcal{H}/\partial f_\alpha^2,\\
\mathcal{H}^{;i}_{,\alpha} &= \partial^2\mathcal{H}/\partial p_i\partial f_\alpha, & \mathcal{H}_{,i\alpha} &= \partial^2\mathcal{H}/\partial x_i\partial f_\alpha.
\end{aligned} \tag{33}$$

They are again considered as functions of γ_3, and are taken for $f_\alpha = 0$.

Now we briefly describe the algorithm to determine the perturbation expansions (27) and (28) for $\tau(x_n, f_\alpha)$ and for $\tau_{,i}(x_n, f_\alpha)$.

First, we perform ray tracing and compute the ray Ω corresponding to the reference Hamiltonian $\mathcal{H}(x_i, p_j)$ and specified by proper initial conditions. Afterwards, we perform the dynamic ray tracing (14), again for proper initial conditions. The initial conditions for the dynamic ray tracing specified at the initial point of the ray Ω, which may be situated at a point source, at an initial surface or at an initial line, are derived and discussed in detail by Červený & Moser (2009). The dynamic ray tracing may be performed along a known ray Ω or together with the ray tracing. By dynamic ray tracing, we obtain Q_{iK} and P_{iK} along the whole ray Ω. Then we can specify the perturbation Hamiltonian $\mathcal{H}(x_i, p_j, f_\alpha)$ and determine its perturbation derivatives $\mathcal{H}_{,\alpha}, \mathcal{H}_{,\alpha\alpha}, \mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$, necessary for the evaluation of eqs (32). At this moment we can start the computation of the perturbation expansions (27) and (28).

The simplest is the evaluation of the term $\tau_{,\alpha}(x_m)$ in the perturbation expansion of the traveltime $\tau(x_m, f_\alpha)$, see (27). We obtain it from the first equation in (30), in which $T_\alpha(\gamma_3)$ is given by the first equation in (32). We can see that for the determination of $\tau_{,\alpha}$ it is not necessary to perform the dynamic ray tracing and compute Q_{ik} and P_{ik}. Moreover, it is not necessary to compute $\mathcal{H}_{,\alpha\alpha}, \mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$. It is just sufficient to perform quadratures of $-\mathcal{H}_{,\alpha}$ along the ray Ω. This basic result is well known from previous studies.

The next step is to compute the first-order perturbation of the traveltime gradient, $\tau_{,i\alpha}(x_m)$. It can be obtained from eq. (31), in which $T_{k\alpha}(\gamma_3)$ is given in the second equation in (32). In this case, the dynamic ray tracing along the ray Ω must be performed to determine Q_{iK} and P_{iK}. Also the perturbation derivatives $\mathcal{H}_{,\alpha}, \mathcal{H}^{;i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ must be determined along the whole ray. Then we can evaluate the integral in the second equation of (32). Note that $T_{K\alpha}(\gamma_3^0)$ vanishes if the initial point of the ray is situated at a point source. If it is situated at an initial surface or line, the quantity $T_{K\alpha}(\gamma_3^0)$ vanishes only if homogeneous waves are considered at τ_0. The procedure of perturbation Hamiltonian is, however, applicable even if inhomogeneous waves are considered at τ_0. In such a case, the quantity $T_{K\alpha}(\gamma_3^0)$ is non-vanishing and its value is controlled by the inhomogeneity of the considered wave at τ_0.

In the last step, we compute the second perturbation derivative $\tau_{,\alpha\alpha}(x_n)$ and the whole second-order expansion of the traveltime (27). For this purpose, we use the second equation in (30). The term $T_{\alpha\alpha}(\gamma_3)$ is given in the last equation of (32). For the evaluation of $T_{\alpha\alpha}(\gamma_3)$ we need to know the perturbation derivatives of the perturbation Hamiltonian $\mathcal{H}_{,\alpha\alpha}$ and $\mathcal{H}^{\cdot i}_{,\alpha}$ and second derivative $\mathcal{H}^{\cdot ij}$. In addition, we also need to know the perturbation derivatives $\tau_{,i\alpha}$ determined in the previous step. From this reason, the computation of the first-order perturbation derivative of the traveltime gradient $\tau_{,i\alpha}$ must always precede the computation of the second-order perturbation derivative $\tau_{,\alpha\alpha}$.

Note that the expression for $T_{\alpha\alpha}$ in (32) does not explicitly contain Q_{iK} and P_{iK} obtained from the dynamic ray tracing. In general, however, $T_{\alpha\alpha}$ depends on Q_{iK} and P_{iK} through $\tau_{,i\alpha}$.

If b^I_{ijkl} in (17) is non-zero, the perturbed medium is dissipative. The perturbation expansions (27) and (28) for traveltime $\tau(\gamma_3, f_\alpha)$ and traveltime gradient $\tau_{,i}(\gamma_3, f_\alpha)$ are then complex-valued

$$\tau(\gamma_3, f_\alpha) = \mathrm{Re}\tau(\gamma_3, f_\alpha) + \mathrm{i}\mathrm{Im}\tau(\gamma_3, f_\alpha), \tag{34}$$

$$\tau_{,i}(\gamma_3, f_\alpha) = \mathrm{Re}\tau_{,i}(\gamma_3, f_\alpha) + \mathrm{i}\mathrm{Im}\tau_{,i}(\gamma_3, f_\alpha). \tag{35}$$

Note that the real-valued vector $\mathbf{P}$ with components $P_i = \mathrm{Re}\tau_{,i}$ is often called the propagation vector, and the real-valued vector $\mathbf{A}$ with components $A_i = \mathrm{Im}\tau_{,i}$ is called the attenuation vector. The wave for which $\mathbf{P}$ is parallel to $\mathbf{A}$ at a point is usually referred to as homogeneous at that point. The wave, for which $\mathbf{P}$ has a direction different than $\mathbf{A}$, is referred to as inhomogeneous at the given point. This terminology is taken from the theory of inhomogeneous plane waves propagating in lossy media, see Hosten *et al.* (1987), Leroy *et al.* (1988), Declercq *et al.* (2005) and Červený & Pšenčík (2005). For different perturbation Hamiltonians, propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ may be different.

In the perturbed dissipative medium, we are mostly interested in the first-order perturbation expansions of τ and $\tau_{,i}$ only. For this reason, we introduce the terminology related to the first-order perturbation expansion. From (34), (35) and (27), (28), we obtain for $f_\alpha = 1$

$$\mathrm{Im}\tau(\gamma_3, f_\alpha) \approx \mathrm{Im}\tau_{,\alpha}(\gamma_3), \tag{36}$$

$$\mathrm{Im}\tau_{,i}(\gamma_3, f_\alpha) \approx \mathrm{Im}\tau_{,i\alpha}(\gamma_3). \tag{37}$$

Note that the first terms of the RHS of (27) and (28), $\tau(x_m)$ and $\tau_{,i}(x_m)$, are real-valued and thus do not appear in (36) and (37).

In the ray theory of time-harmonic seismic body waves, the time-dependent exponential factor is usually considered in the form $\exp[-\mathrm{i}\omega(t-\tau)]$, where τ is the traveltime along the ray. In dissipative media, the traveltime is complex-valued, and the time-dependent exponential term can be factorized

$$\exp[-\mathrm{i}\omega(t - \mathrm{Re}\tau)]\exp[-\omega\mathrm{Im}\tau]. \tag{38}$$

The second factor, $\exp[-\omega\,\mathrm{Im}\tau]$, is often called the dissipation filter $D(\omega)$. Using (36), we can express it in the following form

$$D(\omega) \approx \exp[-\omega\mathrm{Im}\tau_{,\alpha}(\gamma_3)]. \tag{39}$$

Here $\tau_{,\alpha}(\gamma_3) = T_\alpha(\gamma_3)$, see (30), and $T_\alpha(\gamma_3)$ is given by the first expression in (32).

We now introduce several important quantities and relevant notations related to the dissipative perturbed medium ($b^I_{ijkl} \neq 0$).

The prime above symbols denotes that only the first-order perturbation expansion is considered, and that factor f_α is chosen $f_\alpha = 1$. In the following, we introduce attenuation vector $\overline{\mathbf{A}}$, dissipation filter $\overline{D}(\omega)$, global absorption factor $\overline{t}^*(\gamma_3, \gamma_3^0)$, local quality factor $\overline{Q}$, and local attenuation coefficient $\overline{\alpha}$.

Attenuation vector $\overline{\mathbf{A}}$ is introduced by the relation

$$\overline{A}_i(\gamma_3) = \mathrm{Im}\tau_{,i\alpha}(\gamma_3), \tag{40}$$

where perturbation derivative $\tau_{,i\alpha}(\gamma_3)$ is given by (31). Attenuation vector $\overline{\mathbf{A}}$ was introduced in the above way by Červený *et al.* (2008), who used perturbation Hamiltonian (19). The properties of attenuation vector $\overline{\mathbf{A}}$ in a heterogeneous anisotropic weakly dissipative medium were thoroughly investigated there. The above authors specified their results also for plane waves and for waves generated by a point source in a homogeneous anisotropic weakly dissipative medium. In this paper, we generalize the obtained results also for linear perturbation Hamiltonian (20).

Eqs (31) and (40) yield

$$Q_{ia}(\gamma_3)\overline{A}_i(\gamma_3) = \mathrm{Im}T_{a\alpha}(\gamma_3). \tag{41}$$

Eq. (41) can also be expressed as

$$Q_{iK}(\gamma_3)\overline{A}_i(\gamma_3) = \mathrm{Im}T_{K\alpha}(\gamma_3), \quad Q_{i3}(\gamma_3)\overline{A}_i(\gamma_3) = \mathrm{Im}T_{3\alpha}(\gamma_3). \tag{42}$$

The first equation of (42) shows that the real-valued scalars $\mathrm{Im}T_{1\alpha}(\gamma_3)$ and $\mathrm{Im}T_{2\alpha}(\gamma_3)$ are proportional to the projections of attenuation vector $\overline{\mathbf{A}}(\gamma_3)$ into the plane tangent to the wave front at its intersection with reference ray Ω. If $\mathrm{Im}T_{K\alpha}(\gamma_3)$ are non-zero, the wave is inhomogeneous at point γ_3. If they are zero, the wave is homogeneous. We call $\mathrm{Im}T_{K\alpha}(\gamma_3)$ the inhomogeneity terms. In heterogeneous weakly dissipative media, $\mathrm{Im}T_{1\alpha}(\gamma_3)$ and $\mathrm{Im}T_{2\alpha}(\gamma_3)$ are, in general, non-zero. Consequently, the waves propagating in heterogeneous weakly dissipative media are, in general, inhomogeneous. For waves propagating in anisotropic weakly dissipative media, the exceptions are very rare. Only the waves generated by point sources in isotropic homogeneous weakly dissipative media are homogeneous. The plane waves in homogeneous media may be homogeneous or inhomogeneous, depending on how the initial conditions are chosen.

If we consider the third equation in (32), the second equation of (42) and the relation $Q_{i3} = \mathcal{U}_i$, we get

$$\mathcal{U}_i(\gamma_3)\overline{A}_i(\gamma_3) = \mathrm{Im}T_{3\alpha}(\gamma_3) = -\mathrm{Im}\mathcal{H}_{,\alpha}(\gamma_3). \tag{43}$$

Dissipation filter $\overline{D}(\omega)$ follows from (39)

$$\overline{D}(\omega) = \exp\left[-\frac{1}{2}\omega\overline{t}^*(\gamma_3, \gamma_3^0)\right], \tag{44}$$

where $\overline{t}^*(\gamma_3, \gamma_3^0)$ is given by the relation

$$\overline{t}^*\left(\gamma_3, \gamma_3^0\right) = 2\mathrm{Im}\tau_{,\alpha}(\gamma_3) = -2\mathrm{Im}\int_{\gamma_3^0}^{\gamma_3}\mathcal{H}_{,\alpha}(\gamma_3')\mathrm{d}\gamma_3'. \tag{45}$$

In seismology, quantity $\overline{t}^*(\gamma_3, \gamma_3^0)$ is usually called the global absorption factor (Gajewski & Pšenčík 1992; Červený 2001), or the t-star factor. The global absorption factor $\overline{t}^*(\gamma_3, \gamma_3^0)$ characterizes the complete dissipative decay of amplitudes of seismic body waves propagating in a heterogeneous anisotropic (or isotropic) weakly dissipative medium along reference ray Ω from γ_3^0 to γ_3. The dimension of $\overline{t}^*$ is time.

As in isotropic media, the local quality factor $\overline{Q}(\gamma_3)$ is defined by its relation to $\overline{t}^*(\gamma_3, \gamma_3^0)$

$$\overline{t}^*\left(\gamma_3, \gamma_3^0\right) = \int_{\gamma_3^0}^{\gamma_3}\frac{1}{\overline{Q}(\gamma_3')}\mathrm{d}\gamma_3'. \tag{46}$$

Comparison of (45) with (46) then yields an alternative definition of $\overline{Q}(\gamma_3)$

$$1/\overline{Q}(\gamma_3) = 2\mathrm{Im}T_{3\alpha}(\gamma_3) = -2\mathrm{Im}\mathcal{H}_{,\alpha}(\gamma_3). \tag{47}$$

Here $\mathcal{H}_{,\alpha}(\gamma_3)$ is the perturbation derivative of perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$. As it is extremely simple to determine $\mathcal{H}_{,\alpha}$ from $\mathcal{H}(x_a, p_b, f_\alpha)$, the computation of $\overline{Q}(\gamma_3)$ does not cause any difficulty. $\overline{Q}(\gamma_3)$ is a local quantity, which does not require any integration along reference ray Ω.

The local quality factor $\overline{Q}(\gamma_3)$ is a positive, dimensionless real-valued scalar quantity. In an isotropic medium, it is direction-independent. In an anisotropic medium, however, it depends on the direction of wave propagation, specified by ray-velocity vector $\boldsymbol{\mathcal{U}}$ (tangent to reference ray Ω).

Alternatively, we can use a new integration variable s' along ray Ω in (46), representing the arclength, $\mathrm{d}\gamma_3' = \mathrm{d}s'/\mathcal{U}$, and obtain

$$\overline{t}^*(s, s_0) = \int_{s_0}^{s} \frac{1}{\overline{Q}(s')\mathcal{U}(s')}\mathrm{d}s'. \tag{48}$$

Global absorption factor $\overline{t}^*$ and local quality factor $\overline{Q}$ do not depend on inhomogeneity terms $\mathrm{Im}T_{K\alpha}(\gamma_3)$. Consequently, in weakly dissipative media, they do not depend on the inhomogeneity of the wave under consideration. Using (47) with (43), we obtain an alternative expression for $\overline{Q}(\gamma_3)$

$$1/\overline{Q}(\gamma_3) = 2\mathcal{U}_i(\gamma_3)\overline{A}_i(\gamma_3). \tag{49}$$

The multiplication of $\overline{\mathbf{A}}$ by $\boldsymbol{\mathcal{U}}$ cancels the inhomogeneity terms $\mathrm{Im}T_{K\alpha}(\gamma_3)$. Consequently, quantities $\overline{t}^*$ and $\overline{Q}$ are not influenced by the inhomogeneity of the wave under consideration, and represent convenient measures of intrinsic dissipative properties of rocks in the ray direction.

As a simple example, let us consider perturbation Hamiltonian (19). The local quality factor $\overline{Q}(\gamma_3)$ in a heterogeneous anisotropic weakly dissipative medium then reads

$$1/\overline{Q}(\gamma_3) = b^I_{ijkl}\, p_j p_l g_i g_k. \tag{50}$$

Expression (50) for the local quality factor $\overline{Q}$ in a heterogeneous anisotropic weakly dissipative medium has already been derived in a different way by Gajewski & Pšenčík (1992), see also Thomson (1997), Červený (2001, section 5.5) and Červený *et al.* (2008). For electromagnetic waves, a similar equation was given by Lewis (1965). Eqs (49) and (50) were also derived, in a different way, for plane waves propagating in homogeneous weakly dissipative anisotropic media by Červený & Pšenčík (2008).

Different forms of perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ yield different expressions for the local quality factor $\overline{Q}$, which differ in accuracy. See the more detailed study for isotropic media in Section 5.2.

Another quantity, often used in seismology to specify the dissipative properties of rocks, is the local attenuation coefficient $\overline{\alpha}(s)$. The expression for $\overline{\alpha}(s)$ follows from the expression for dissipation filter (44), in which we use (48) for $\overline{t}^*(s, s_0)$. We can then express the dissipation filter in the following way

$$\overline{D}(\omega) = \exp\left[-\int_{s_0}^{s}\overline{\alpha}(s')\mathrm{d}s'\right], \tag{51}$$

where $\overline{\alpha}(s)$ is given by the relation

$$\overline{\alpha}(s) = \omega\mathrm{Im}\mathcal{H}_{,\alpha}(s)/\mathcal{U}(s) = \omega/2\overline{Q}(s)\mathcal{U}(s). \tag{52}$$

In these expressions, arclength s is used along reference ray Ω instead of γ_3. The relation for attenuation coefficient $\overline{\alpha}(s) = \omega/2\overline{Q}(s)\mathcal{U}(s)$ also holds for plane waves, see Červený & Pšenčík (2008).

4 THE PERTURBATION EXPANSION IN TERMS OF THE LINEAR PERTURBATION HAMILTONIAN

In this section, we consider the linear perturbation Hamiltonian introduced by (20). We use it to derive the second-order perturbation expansion for traveltime $\tau(\gamma_3, f_\alpha)$ and the first-order perturbation expansions for traveltime gradient $\tau_{,i}(\gamma_3, f_\alpha)$. The reference ray is constructed in a reference, perfectly elastic, isotropic or anisotropic, heterogeneous medium. The general perturbation expansions for these quantities for an arbitrary form of perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ are presented in Section 3.

We remind the reader that vector $\tilde{\mathbf{g}}$ is introduced as the eigenvector of the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ given by (24). It is real-valued if the perturbed medium is perfectly elastic, and complex-valued if the perturbed medium is dissipative. The Hamiltonians $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are introduced by (22) and (23). Hamiltonian $\tilde{\mathcal{H}}$ can be simply computed at any point of the reference ray once eigenvector $\tilde{\mathbf{g}}$ is determined.

The perturbation derivatives (33) of linear perturbation Hamiltonian (20) are then given by extremely simple expressions

$$\mathcal{H}_{,\alpha} = \tilde{\mathcal{H}} - \mathcal{H}^0, \qquad \mathcal{H}^{,i}_{,\alpha} = \tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i},$$
$$\mathcal{H}_{,\alpha\alpha} = 0, \qquad \mathcal{H}_{,i\alpha} = \tilde{\mathcal{H}}_{,i} - \mathcal{H}^0_{,i}. \tag{53}$$

The perturbation derivatives of the traveltime and of the traveltime gradient are given by (30) and (31), where $T_\alpha(\gamma_3)$, $T_{k\alpha}(\gamma_3)$ and $T_{\alpha\alpha}(\gamma_3)$ are given by relations

$$T_\alpha(\gamma_3) = -\int_{\gamma_3^0}^{\gamma_3}(\tilde{\mathcal{H}} - \mathcal{H}^0)\mathrm{d}\gamma_3',$$
$$T_{K\alpha}(\gamma_3) = T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3}\left[(\tilde{\mathcal{H}}_{,i} - \mathcal{H}^0_{,i})Q_{iK} - (\tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i})P_{iK}\right]\mathrm{d}\gamma_3',$$
$$T_{3\alpha}(\gamma_3) = \mathcal{H}^0 - \tilde{\mathcal{H}},$$
$$T_{\alpha\alpha}(\gamma_3) = -\int_{\gamma_3^0}^{\gamma_3}\tau_{,i\alpha}\left[2(\tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i}) + \mathcal{H}^{,ij}\tau_{,j\alpha}\right]\mathrm{d}\gamma_3'. \tag{54}$$

Inserting (30) and (31) with (54) into perturbation expansions (27) and (28), where $f_\alpha = 1$, we obtain the final solution of our problem. The resulting equations are valid quite generally, for isotropic and anisotropic, homogeneous and heterogeneous, perfectly elastic and weakly dissipative, media.

For real-valued perturbations ($b^I_{ijkl} = 0$), linear perturbation Hamiltonian (20) and the expressions for the first- and second-order perturbations of the traveltimes were used in studying the coupling ray theory of S waves by Klimeš & Bulant (2004, 2006) and by Bulant & Klimeš (2008). The first- and second-order perturbations of traveltimes in anisotropic media were computed along reference rays in isotropic reference media by Klimeš & Bulant (2004). Klimeš & Bulant (2006) extended the approach to the common S-wave reference ray in a reference anisotropic medium. Bulant & Klimeš (2008) discussed numerical examples and the accuracy of the approach. Klimeš (2006a) also presented an algorithm for computing amplitudes using the general perturbation Hamiltonian.

The studies of Klimeš & Bulant (2004, 2006) suggest that the method based on the linear perturbation Hamiltonian with $N = -1$ is highly accurate and universal, at least for $b^I_{ijkl} = 0$. Its accuracy clearly exceeds the accuracy of the method based on perturbation Hamiltonian (19). Even if we use only the first-order perturbation derivative $\tau_{,\alpha}$, the accuracy highly exceeds the accuracy of standard perturbation methods. The reason consists mainly in the exact computation of the eigenvector $\tilde{\mathbf{g}}$ of the perturbed Christoffel matrix $\tilde{\Gamma}(x_a, p_b)$, see (24). Another reason for expecting good results for $N = -1$ is as follows. If the ray trajectory is not changed during the perturbation, the method based on the linear perturbation Hamiltonian with $N = -1$ leads to exact results. Other choices of N yield approximate results only.

The method based on the linear perturbation Hamiltonian, applied to complex-valued perturbations b_{ijkl}, corresponds to weakly dissipative media. Particularly for $N = -1$, it is expected to yield a higher accuracy than the other perturbation methods. Study of accuracy of the method for weakly dissipative media requires detailed numerical tests.

We now consider weakly dissipative media ($b^I_{ijkl} \neq 0$), and summarize the relevant expressions for the quantities discussed in Section 3, based on the linear perturbation Hamiltonian (20).

The attenuation vector $\overline{\mathbf{A}}$, introduced in (40), is given by relations (taking into account that $Q^{-1}_{3i} = p_i$)

$$\overline{A}_i(\gamma_3) = \mathrm{Im}\tau_{,i\alpha}(\gamma_3) = Q^{-1}_{Ki}(\gamma_3)\mathrm{Im}T_{K\alpha}(\gamma_3) - p_i\mathrm{Im}\tilde{\mathcal{H}}(\gamma_3). \tag{55}$$

Here $T_{K\alpha}(\gamma_3)$ is given by the second expression in (54). Dissipation filter $\overline{D}(\omega)$ is given by (44), where the global absorption factor $\overline{t}^*(\gamma_3, \gamma_3^0)$ is given by the relation

$$\overline{t}^*(\gamma_3, \gamma_3^0) = -2\mathrm{Im}\int_{\gamma_3^0}^{\gamma_3} \tilde{\mathcal{H}}(\gamma_3')\mathrm{d}\gamma_3', \tag{56}$$

where $\tilde{\mathcal{H}}(\gamma_3')$ is given by (23). The local quality factor $\overline{Q}(\gamma_3)$, defined by (47), is now expressed as follows

$$1/\overline{Q}(\gamma_3) = -2\mathrm{Im}\tilde{\mathcal{H}}(\gamma_3). \tag{57}$$

Finally, the local attenuation coefficient $\overline{\alpha}(s)$, defined in (52) is given by the relation

$$\overline{\alpha}(s) = \frac{\omega\mathrm{Im}\tilde{\mathcal{H}}(s)}{\mathcal{U}(s)} = \frac{\omega}{2\overline{Q}(s)\mathcal{U}(s)}. \tag{58}$$

5 SPECIAL CASES

In this section, we specify the general equations presented in Section 4, based on linear perturbation Hamiltonian (20), in two special cases.

5.1 Perturbation from perfectly elastic to weakly dissipative media

We consider the perturbed medium specified by density normalized viscoelastic moduli

$$a_{ijkl} - \mathrm{i}b^I_{ijkl}. \tag{59}$$

Consequently, $b^R_{ijkl} = 0$. The real-valued parts a_{ijkl} of the density-normalized viscoelastic moduli in the perturbed medium equal density-normalized elastic moduli in the reference medium; but their imaginary parts b^I_{ijkl} (dissipation moduli) are non-vanishing only in the perturbed media.

This case was studied in detail by Červený *et al.* (2008), using perturbation Hamiltonian (19). Here we shall use the linear perturbation Hamiltonian (20). We are not going to derive all the equations. Actually, all the equations can be simply obtained from those presented in Section 4, by inserting $b^R_{ijkl} = 0$. In the following, we discuss only the first-order perturbations of traveltime $\tau_{,\alpha}(\gamma_3)$, which will clearly show the differences between the approaches based on the perturbation Hamiltonians (19) and (20).

Perturbation Hamiltonian (19) used in (30) and (32) yields

$$\tau_{,\alpha}(\gamma_3) = \frac{\mathrm{i}}{2}\int_{\gamma_3^0}^{\gamma_3} b^I_{ijkl}p_j p_l g_i g_k \mathrm{d}\gamma_3'. \tag{60}$$

Consequently, $\tau_{,\alpha}(\gamma_3)$ is purely imaginary. It causes a decay of amplitudes. It does not yield any change in the real-valued traveltime.

Using linear perturbation Hamiltonian (20), (30) and (54) yield

$$\tau_{,\alpha}(\gamma_3) = -\int_{\gamma_3^0}^{\gamma_3} (\tilde{\mathcal{H}} - \mathcal{H}^0)\mathrm{d}\gamma_3', \tag{61}$$

where $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are given by (22) and (23). In (23), $\tilde{\mathbf{g}}$ is an exact eigenvector of the perturbed Christoffel matrix (24). Expression (61) yields more accurate results than expression (60), because it uses exact eigenvectors of the complex-valued Christoffel matrix (24) while (60) uses only their perturbation expansion. The important difference between (60) and (61) is that $\tau_{,\alpha}(\gamma_3)$ in (60) is purely imaginary while $\tau_{,\alpha}(\gamma_3)$ in (61) is complex-valued. As (61) is complex-valued, the introduced perturbation implies even corrections of real-valued traveltimes. These real-valued traveltime corrections are given by relation

$$\mathrm{Re}(\tau_{,\alpha}(\gamma_3)) = -\mathrm{Re}\int_{\gamma_3^0}^{\gamma_3} (\tilde{\mathcal{H}} - \mathcal{H}^0)\mathrm{d}\gamma_3'. \tag{62}$$

It is interesting that these traveltime corrections due to dissipation are obtained within the framework of the first-order perturbations.

Using the linear perturbation Hamiltonian (20) in the relation (47), we can obtain a more accurate estimation of local quality factor $\overline{Q}$ than that given in (50)

$$1/\overline{Q}(\gamma_3) = -2\mathrm{Im}\tilde{\mathcal{H}}(\gamma_3). \tag{63}$$

It can be clearly seen that (63) would yield (50) if $N = 2$ and if $\tilde{\mathbf{g}} = \mathbf{g}$.

5.2 Isotropic reference model

Perturbation Hamiltonians (19) and (20) can also be used if the reference medium is isotropic perfectly elastic, and the perturbed medium weakly anisotropic and/or weakly dissipative. For the perfectly elastic isotropic reference medium, the density normalized elastic moduli are given by the relation,

$$a_{ijkl} = \left(V_P^2 - 2V_S^2\right)\delta_{ij}\delta_{kl} + V_S^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{64}$$

where V_P and V_S are the velocities of P and S waves in the perfectly elastic isotropic reference medium. The reference Christoffel matrix in the reference medium then reads

$$\Gamma_{ik}(x_a, p_b) = \left(V_P^2 - V_S^2\right)p_i p_k + \delta_{ik}V_S^2 p_l p_l. \tag{65}$$

The eigenvalues of the Christoffel matrix (65) are given by the relation

$$G(x_a, p_b) = V^2(x_n)p_l p_l. \tag{66}$$

For P waves, $V = V_P$, and the eigenvector is a unit vector tangent to slowness vector $\mathbf{p}$ (perpendicular to the wave front). For S waves, $V = V_S$, and the eigenvectors are two mutually perpendicular unit vectors, perpendicular to slowness vector $\mathbf{p}$. The reference Hamiltonian, used in the ray tracing and dynamic ray tracing in a reference medium, is then given by the relation

$$\mathcal{H}^0(x_a, p_b) = \frac{1}{N}\left[V^2(x_a)p_l p_l\right]^{N/2}. \tag{67}$$

Along the reference ray, $G(x_a, p_b)$ and $\mathcal{H}^0(x_a, p_b)$ remain constant, as

$$V^2(x_n)p_l p_l = 1. \tag{68}$$

The derivatives of reference Hamiltonian (67) are then given by relations

$$\mathcal{H}^{0,i} = V^2 p_i, \qquad \mathcal{H}^0_{,i} = \frac{1}{2}(V^2)_{,i}/V^2,$$
$$\mathcal{H}^{0,ij} = 2(\frac{N}{2} - 1)V^4 p_i p_j + V^2\delta_{ij}, \qquad \mathcal{H}^{0,i}_{,j} = \frac{N}{2}(V^2)_{,j}p_i,$$
$$\mathcal{H}^0_{,ij} = \frac{1}{2}(\frac{N}{2} - 1)(V^2)_{,i}(V^2)_{,j}/V^4 + \frac{1}{2}(V^2)_{,ij}/V^2. \tag{69}$$

Here we have also used eq. (68). Note that the ray tracing system (10) in the reference medium does not depend on N.

The homogeneous perturbation Hamiltonian (19) of the second degree ($N = 2$) for isotropic reference medium reads

$$\mathcal{H}(x_a, p_m, f_\alpha) = \frac{1}{2}\left(V^2 p_i p_i + f_\alpha b_{ijkl} p_j p_l \bar{g}_i \bar{g}_k\right), \tag{70}$$

The linear perturbation Hamiltonian (20) of the Nth degree is given by relation (20) with (21) and (23), and with reference Hamiltonian $\mathcal{H}^0(x_j, p_b)$ given by the relation

$$\mathcal{H}^0(x_a, p_b) = \frac{1}{N}\left(V^2 p_k p_k\right)^{N/2}. \tag{71}$$

Eigenvector $\tilde{\mathbf{g}}$ is the eigenvector appropriate to the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$, given by the relation

$$\tilde{\Gamma}_{ik} = \left(V_P^2 - V_S^2\right) p_i p_k + \delta_{ik} V_S^2 p_n p_n + b_{ijkl} p_j p_l. \tag{72}$$

The difference between the first and second form of perturbation Hamiltonians (19) and (20) consists in the use of eigenvectors $\bar{\mathbf{g}}$ and $\tilde{\mathbf{g}}$ and in the freedom to choose the degree of homogeneity N of the Hamiltonian under consideration. Moreover, the perturbation Hamiltonian (19) is non-linear, but (20) is linear, in terms of perturbation parameter f_α.

The perturbations b_{ijkl} in (70) and (72) are defined as the differences between the density-normalized viscoelastic moduli in the perturbed anisotropic medium and the density-normalized elastic moduli a_{ijkl} given by (64) in the reference perfectly elastic isotropic medium. The perturbed medium may be weakly anisotropic and weakly dissipative (b_{ijkl} complex-valued) or weakly anisotropic and perfectly elastic (b_{ijkl} real-valued). For purely imaginary b_{ijkl}, the perturbed medium is characterized by anisotropic attenuation. Imaginary b_{ijkl} may also generate changes of the real-valued traveltime due to dissipation.

The results considerably simplify if also the perturbed medium is isotropic and weakly dissipative. The eigenvalue of the Christoffel matrix in the perturbed medium can then again be expressed in the form of (66), where V^2 is replaced by a complex-valued function $V^2(x_i) + b(x_i)$. Linear perturbation Hamiltonian (20) is then given by equation

$$\mathcal{H}(x_a, p_b, f_\alpha) = \mathcal{H}^0(x_a, p_b) + f_\alpha\left[\tilde{\mathcal{H}}(x_a, p_b) - \mathcal{H}^0(x_a, p_b)\right], \tag{73}$$

where $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are given by expressions

$$\mathcal{H}^0 = \frac{1}{N}\left[V^2 p_k p_k\right]^{N/2}, \quad \tilde{\mathcal{H}} = \frac{1}{N}\left[(V^2 + b)p_k p_k\right]^{N/2}. \tag{74}$$

The relations for $\tau_{,\alpha}(\gamma_3)$, $\tau_{,i\alpha}(\gamma_3)$ and $\tau_{,\alpha\alpha}(\gamma_3)$ remain the same as in (30) and (31) with (54), where we use

$$\mathcal{H}^{0,i} = V^2 p_i, \qquad \tilde{\mathcal{H}}^{,i} = (1 + bV^{-2})^{N/2-1}(V^2 + b)p_i,$$
$$\mathcal{H}^0_{,i} = \frac{1}{2}(V^2)_{,i} p_n p_n, \quad \tilde{\mathcal{H}}_{,i} = \frac{1}{2}(1 + bV^{-2})^{N/2-1}(V^2 + b)_{,i} p_n p_n. \tag{75}$$

Note that the resulting expressions are not influenced by the eigenvectors of the Christoffel matrices. Consequently, the differences between $\tilde{\mathbf{g}}$ and $\bar{\mathbf{g}}$ play no role in isotropic media.

Let us now discuss in greater detail the most important expression for $\tau_{,\alpha}(\gamma_3)$. From (30) with (54) and (75), we obtain a simple expression for $\tau_{,\alpha}(\gamma_3)$, valid for any N

$$\tau_{,\alpha}(\gamma_3) = -\frac{1}{N}\int_{\gamma_3^0}^{\gamma_3}[(1 + bV^{-2})^{N/2} - 1]\mathrm{d}\gamma_3'. \tag{76}$$

In this and the following integrals, V and b are functions of spatial coordinates x_i, $V = V(x_i)$ and $b = b(x_i)$ along the reference ray. Alternatively, they can be understood as functions of γ_3' as γ_3' determines strictly positions of points along the ray.

For the commonly used case of $N = 2$, which corresponds to the perturbation Hamiltonian (19) in isotropic media, expression (76) simplifies to

$$\tau_{,\alpha}(\gamma_3) = -\frac{1}{2}\int_{\gamma_3^0}^{\gamma_3} bV^{-2}\mathrm{d}\gamma_3'. \tag{77}$$

For real-valued b, formula (77) corresponds to the well-known traveltime perturbation formula for isotropic inhomogeneous media. For example, if we use notation $b = \delta V^2 = 2V\,\delta V$ and take into account $\mathrm{d}\gamma_3' = \mathrm{d}s/V$, we obtain the equation given by Aki & Richards (1980, p. 797, Problems 13.3). Another important case is $N = -1$

$$\tau_{,\alpha}(\gamma_3) = \int_{\gamma_3^0}^{\gamma_3}[(1 + bV^{-2})^{-1/2} - 1]\mathrm{d}\gamma_3'. \tag{78}$$

It is simple to see that (76) and (78) approach (77) for $bV^{-2} \to 0$. In general, however, the accuracy of the expression (76) for $\tau_{,\alpha}(\gamma_3)$ depends on N. It was shown numerically by Bulant & Klimeš (2008) that the most accurate results for real-valued b are obtained for $N = -1$.

To make the expressions for $\tau_{,\alpha}(\gamma_3)$ more transparent, we express perturbation term b in terms of Lamé's viscoelastic complex-valued moduli λ and μ. According to the definition, b is the perturbation of the density-normalized elastic modulus corresponding to the wave under consideration. It is possible to discuss b independently for P and S waves. For brevity, however, we consider only P waves; for S waves the results would be quite analogous. The exact relation between the density normalized viscoelastic complex-valued modulus of a P wave and complex-valued phase velocity α_P in a dissipative isotropic medium reads

$$\frac{\lambda + 2\mu}{\rho} = \alpha_P^2 = \mathrm{Re}\,\alpha_P^2 + \mathrm{i}\,\mathrm{Im}\,\alpha_P^2. \tag{79}$$

In the reference, perfectly elastic medium, λ, μ and α_P are real-valued. Then $\alpha_P^2 = \mathrm{Re}(\lambda + 2\mu)/\rho = V_P^2$, where V_P is the real-valued velocity of P waves, see (64). In the perturbed medium we mark the quantities α_P, λ, μ and ρ with a tilde above the letter. We obtain

$$\tilde{\alpha}_P^2 = \mathrm{Re}\frac{\tilde{\lambda} + 2\tilde{\mu}}{\tilde{\rho}} + \mathrm{i}\,\mathrm{Im}\frac{\tilde{\lambda} + 2\tilde{\mu}}{\tilde{\rho}} = \tilde{V}_P^2(1 - \mathrm{i}\tilde{\delta}), \tag{80}$$

where

$$\tilde{\delta} = -\mathrm{Im}(\tilde{\lambda} + 2\tilde{\mu})/\mathrm{Re}(\tilde{\lambda} + 2\tilde{\mu}). \tag{81}$$

Symbol $\tilde{V}_P$ thus denotes the real-valued P-wave velocity in the perturbed medium. Consequently, perturbation term b is given by the relation

$$b = \tilde{\alpha}_P^2 - V_P^2 = \tilde{V}_P^2(1 - \mathrm{i}\tilde{\delta}) - V_P^2. \tag{82}$$

From (82), we obtain

$$V_P^2 + b = \tilde{V}_P^2(1 - \mathrm{i}\tilde{\delta}). \tag{83}$$

Inserting (83) into (76)–(78), we obtain for any N

$$\tau_{,\alpha}(\gamma_3) = -\frac{1}{N}\int_{\gamma_3^0}^{\gamma_3} \left\{\left[(1 - \mathrm{i}\tilde{\delta})\tilde{V}_P^2/V_P^2\right]^{N/2} - 1\right\} \mathrm{d}\gamma_3', \tag{84}$$

for $N = 2$,

$$\tau_{,\alpha}(\gamma_3) = -\frac{1}{2}\int_{\gamma_3^0}^{\gamma_3} \left[(1 - \mathrm{i}\tilde{\delta})\tilde{V}_P^2/V_P^2 - 1\right] \mathrm{d}\gamma_3', \tag{85}$$

and for $N = -1$,

$$\tau_{,\alpha}(\gamma_3) = \int_{\gamma_3^0}^{\gamma_3} \left(\frac{1}{\tilde{V}_P\sqrt{1 - \mathrm{i}\tilde{\delta}}} - \frac{1}{V_P}\right) V_P \mathrm{d}\gamma_3'. \tag{86}$$

Note that $V_P\, \mathrm{d}\gamma_3' = \mathrm{d}s$, where s is the arclength along the ray in the reference medium. Thus, the integration variable in (86) may be substituted by the arclength along the ray.

Let us now determine local quality factor $\overline{Q}(\gamma_3)$. For arbitrary N, we obtain it from (84) by removing the integration sign and taking twice its imaginary part, see (61) and (63). As V_P is real-valued, we obtain

$$1/\overline{Q} = -\frac{2}{N}\mathrm{Im}\left[(1 - \mathrm{i}\tilde{\delta})(\tilde{V}_P/V_P)^2\right]^{N/2}. \tag{87}$$

The expression (87) for $\overline{Q}$ can also be expressed as follows

$$1/\overline{Q} = -\frac{2}{N}\mathrm{Im}\left[(1 - \mathrm{i}\tilde{\delta})\left(1 + \frac{\tilde{V}_P^2 - V_P^2}{V_P^2}\right)\right]^{N/2}. \tag{88}$$

This is a very general expression, valid for any N, and containing both structural perturbation $(\tilde{V}_P^2 - V_P^2)/V_P^2$, and dissipative perturbation $\tilde{\delta}$.

If both these perturbations are small enough, we can linearize the term inside the square brackets in (88), and obtain,

$$1/\overline{Q} \approx -\frac{2}{N}\mathrm{Im}\left[1 - \mathrm{i}\tilde{\delta} + \left(\tilde{V}_P^2 - V_P^2\right)/V_P^2\right]^{N/2}. \tag{89}$$

The first-order linear expansion of (89) then yields

$$1/\overline{Q} \approx \mathrm{Im}\left[\mathrm{i}\tilde{\delta} - \left(\tilde{V}_P^2 - V_P^2\right)/V_P^2\right]. \tag{90}$$

Finally,

$$1/\overline{Q} \approx \tilde{\delta} = -\mathrm{Im}(\tilde{\lambda} + 2\tilde{\mu})/\mathrm{Re}(\tilde{\lambda} + 2\tilde{\mu}) = -\mathrm{Im}\tilde{\alpha}_P^2/\mathrm{Re}\tilde{\alpha}_P^2. \tag{91}$$

Thus, the linearization of (88) yields a simple expression (91) for $1/\overline{Q}$, independent of the structural perturbations and of N.

In homogeneous viscoelastic isotropic media, the quality factor of plane waves has mostly been defined by the well-known relation, corresponding to (91) (Aki & Richards 1980; Johnston & Toksöz 1981, etc.). Thus, the linearization of (88) yields the same relation for $1/\overline{Q}$, valid even for any seismic body waves, propagating in inhomogeneous, isotropic, weakly dissipative media. The general expression (88), however, also predicts some higher-order effects resulting from the coupling of dissipative and structural perturbations in inhomogeneous media.

6 CONCLUSIONS

Perturbation Hamiltonians play an important role in computing perturbations of any order of the traveltime and perturbations of the partial derivatives of the traveltime of any order. The traveltime perturbations under consideration may be complex-valued. This is very important, as also weakly dissipative media may be considered to be small perturbations of perfectly elastic media. The application of perturbation Hamiltonians does not require ray tracing of perturbed rays, and is fully based on the quadratures along the reference real-valued rays in reference elastic media. Consequently, complex-valued ray tracing in a dissipative medium is not required.

Various forms of perturbation Hamiltonians can be used, differing in the accuracy and numerical efficiency of computations. In this paper, a suitable perturbation Hamiltonian (20) is used, referred to as the linear perturbation Hamiltonian. This Hamiltonian offers simple expressions for perturbation derivatives, simple algorithms and high-accuracy computations. We have explicitly derived the expressions for the first- and second-order perturbations of complex-valued traveltimes, and for the first-order perturbations of the traveltime gradient. These perturbations play the most important role in practical applications. The general expressions for attenuation vector $\overline{\mathbf{A}}$, for the local quality factor $\overline{Q}$, for global absorption factor $\overline{t}^*$ and for attenuation coefficient $\overline{\alpha}$ are given for heterogeneous weakly dissipative perturbed media.

For the weakly dissipative perturbed medium, with the same real parts of the density-normalized elastic moduli as in the reference medium, the model perturbations are purely imaginary. Even in this case, the approach based on the linear perturbation Hamiltonian also predicts the perturbations of real-valued traveltimes. Consequently, we obtain the traveltime correction caused by dissipation. The relevant traveltime corrections are involved both in the first-order, but also in the second order traveltime perturbations.

All equations are valid for any heterogeneous, isotropic or anisotropic, perfectly elastic reference media, and for heterogeneous, isotropic or anisotropic, perfectly elastic or weakly dissipative perturbed media. For the isotropic reference medium, the perturbed medium may be isotropic perfectly elastic, isotropic weakly dissipative, weakly anisotropic perfectly elastic, or weakly anisotropic weakly dissipative. Thus, the effects of weak anisotropy and weak dissipation may be treated together.

In our future investigations, we wish to concentrate on numerical studies of the accuracy of the equations for heterogeneous anisotropic weakly dissipative media based on the use of the linear perturbation Hamiltonian. Particularly, we wish to consider the homogeneous linear perturbation Hamiltonian of the order $N = -1$, for which Bulant & Klimeš (2008) obtained highly accurate results for perfectly elastic media. Further, we wish to derive approximate expressions for the high-frequency ray theory Green function in heterogeneous anisotropic weakly dissipative media.

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