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Quality factor Q in dissipative anisotropic media

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ABSTRACT

In an isotropic dissipative medium, the attenuation properties of rocks are usually specified by quality factor Q, a positive, dimensionless, real-valued, scalar quantity, independent of the direction of wave propagation. We propose a similar, scalar, but direction-dependent quality Q-factor (also called Q) for time-harmonic, homogeneous or inhomogeneous plane waves propagating in unbounded homogeneous dissipative anisotropic media. We define the Q-factor, as in isotropic viscoelastic media, as the ratio of the time-averaged complete stored energy and the dissipated energy, per unit volume. A solution of an algebraic equation of the sixth degree with complex-valued coefficients is necessary for the exact determination of Q. For weakly inhomogeneous plane waves propagating in arbitrarily anisotropic, weakly dissipative media, we simplify the exact expression for Q considerably using the perturbation method. The solution of the equation of the sixth degree is no longer required. We obtain a simple, explicit perturbation expression for the quality factor, denoted as $\hat{Q}$. We prove that the direction-dependent $\hat{Q}$ is related to the attenuation coefficient α measured along a profile in the direction of the energy-velocity vector (ray direction). The quality factor $\hat{Q}$ does not depend on the inhomogeneity of the plane wave under consideration and thus is a convenient measure of the intrinsic dissipative properties of rocks in the ray direction. In all other directions, the quality factor is influenced by the inhomogeneity of the wave under consideration. We illustrate the peculiarities in the behavior of $\hat{Q}$ and its accuracy on a model of anisotropic, weakly dissipative sedimentary rock. Examples show interesting inner loops in polar diagrams of $\hat{Q}$ in regions of S-wave triplications.

INTRODUCTION

The dissipative properties of rocks are commonly specified by the quality factor Q (also called Q). In isotropic dissipative media, Q is a positive, dimensionless, scalar quantity, independent of the direction of wave propagation. A review of definitions, terminology and observations in seismology and seismic exploration devoted to isotropic dissipative media can be found in Aki and Richards (1980), Johnston and Toksöz (1981), Cormier (1989), and Carcione (2001, 2006, 2007). The most common definition of the Q-factor of time-harmonic plane P- and S-waves propagating in homogeneous, weakly dissipative (low-loss) isotropic media is $Q = -\mathrm{Re}(M)/\mathrm{Im}(M)$, where M is the complex-valued P- or S-wave viscoelastic modulus. Alternatively, the definition may take the form $Q = -\mathrm{Re}(v_c^2)/\mathrm{Im}(v_c^2)$, where v_c is the complex-valued P- or S-wave phase velocity (e.g., Johnston and Toksöz, 1981; Moczo et al., 1987). The signs in the expressions for Q depend on the sign convention used for time-harmonic waves. With the sign convention used in this paper [the exponential time factor $\exp(-\mathrm{i}\omega t)$], $\mathrm{Im}(M)$ and $\mathrm{Im}(v_c^2)$ are negative; thus, expressions for Q contain negative signs.

We are interested in the direction-dependent Q-factor for anisotropic dissipative media. We consider only those dissipation mechanisms that can be described within the framework of linear viscoelasticity. Classically, the Q-factor of time-harmonic plane waves has been defined by the relation $Q^{-1} = W_d/E$ (Buchen, 1971), where W_d is the density of the time-averaged dissipated energy and E is the density of the time-averaged complete stored energy, $E = K + U$. Quantities K and U are densities of the time-averaged kinetic and strain energies, respectively. For perfectly elastic media, U equals K; but for viscoelastic media, U generally differs from K. Modified versions of the definition $Q^{-1} = W_d/E$ sometimes have been used in seismic literature. For example, Carcione (2001, p. 58–59) uses the definition $Q^{-1} = W_d/2U$. This definition is adopted by Červený and Pšenčík (2006) and by Zhu and Tsvankin (2006, 2007). Here, we consider systematically the classical definition $Q^{-1} = W_d/E$ of Buchen (1971). The definition is applicable to homoge-

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neous and inhomogeneous plane waves propagating in isotropic as well as anisotropic viscoelastic media.

For time-harmonic plane waves propagating in homogeneous, anisotropic, viscoelastic media, the Q-factor has often been introduced in the same way as for isotropic media, using the definition $Q = -\mathrm{Re}(v_c^2)/\mathrm{Im}(v_c^2)$, where v_c is the complex-valued phase velocity of the plane wave under consideration. Individual approaches to determining the Q-factor, based on this definition, differ in the way of computing complex-valued phase velocities, in the specification of the complex-valued slowness vector, and in the treatment of inhomogeneous waves (e.g., Carcione, 1994, 2001, 2007; Krebes and Le, 1994; Chichinina et al., 2006). However, the applicability of this definition of Q to anisotropic media is problematic, particularly for inhomogeneous plane waves. For example, Krebes and Le (1994) show that this definition of Q "gives a generally unreliable measure of anelasticity" for inhomogeneous plane waves propagating in anisotropic viscoelastic media. This might, however, be the consequence of inappropriate parameterization of the slowness vectors of inhomogeneous plane waves using the attenuation angle.

Inhomogeneous waves play an important role in the propagation of waves in viscoelastic isotropic or anisotropic media. The slowness vector $\mathbf{p}$ of waves propagating in viscoelastic media is always complex valued: $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, where $\mathbf{P}$ is the propagation vector, $\mathbf{A}$ the attenuation vector, and i the imaginary unit. For $\mathbf{A}$ parallel to $\mathbf{P}$ or for $\mathbf{A} = \mathbf{0}$, the wave is homogeneous; otherwise, it is inhomogeneous. The advantage of plane waves propagating in homogeneous viscoelastic media is that their inhomogeneity may be chosen arbitrarily. This is not the case for waves generated by a point source in a viscoelastic medium. The inhomogeneity of these waves results from the solution of the problem under consideration. In most cases, including homogeneous anisotropic viscoelastic media, waves propagating from a point source to a receiver are inhomogeneous (Červený et al., 2008). Waves generated by a point source are homogeneous only in unbounded homogeneous isotropic viscoelastic media. In general, the waves propagating in viscoelastic anisotropic media are inhomogeneous, although their inhomogeneity may be weak. We show that the attenuation of the wave does not depend on the strength of its inhomogeneity only for a certain privileged direction, the direction of energy flux.

Inhomogeneous waves play an important role in many branches of physics. Declercq et al. (2005) give an excellent review of ultrasonic inhomogeneous waves, their generation, and applications in nondestructive testing. Applications of inhomogeneous waves in seismology and acoustics are studied by Borcherdt and Wennerberg (1985), Borcherdt et al. (1986), and Hosten et al. (1987).

In viscoelastic media, the stiffnesses c_{ijkl}, relating the stress and strain tensors, are not real valued as in perfectly elastic media but are complex valued. We call them viscoelastic moduli. The imaginary parts of viscoelastic moduli are responsible for the dissipative properties of waves, propagating in viscoelastic media. The sign convention for the imaginary parts of the viscoelastic moduli must correspond to the sign convention used to describe the time-harmonic wave under consideration — particularly to the sign in the exponential time factor $\exp(\pm\mathrm{i}\omega t)$. We use the exponential time factor $\exp(-\mathrm{i}\omega t)$, for which $c_{ijkl} = c_{ijkl}^R - \mathrm{i}c_{ijkl}^I$. To simplify certain relations, we use the density-normalized viscoelastic moduli $a_{ijkl} = c_{ijkl}/\rho$:

$$a_{ijkl} = a_{ijkl}^R - \mathrm{i}a_{ijkl}^I. \tag{1}$$

The density-normalized viscoelastic moduli a_{ijkl} are frequency dependent in viscoelastic media, but we do not write $a_{ijkl}(\omega)$ because we consider an arbitrary but fixed $\omega > 0$. We assume that a_{ijkl} satisfy the symmetry relations

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}. \tag{2}$$

Because of the symmetry relations 2, the number of density-normalized viscoelastic moduli reduces from 81 to 21. They can be described by the 6×6 symmetric matrix in Voigt notation, $\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I$, with elements $\bar{A}_{\alpha\beta}(\alpha,\beta = 1,2,\ldots,6)$. The elements can be expressed as $\bar{A}_{\alpha\beta} = \bar{A}_{\alpha\beta}^R - \mathrm{i}\bar{A}_{\alpha\beta}^I$. We use bars in $\bar{\mathbf{A}}$, $\bar{\mathbf{A}}^R$, and $\bar{\mathbf{A}}^I$ to distinguish the matrix $\bar{\mathbf{A}}$ from attenuation vector $\mathbf{A}$. Hereinafter, we call a_{ijkl}^I and $\bar{A}_{\alpha\beta}^I$ the dissipation moduli, and we call the 6×6 matrix $\bar{\mathbf{A}}^I$ the dissipation matrix. We do not use the so-called $\mathbf{Q}$ matrix with elements $Q_{\alpha\beta} = \bar{A}_{\alpha\beta}^R/\bar{A}_{\alpha\beta}^I$, which is sometimes seen in the literature. The direct application of dissipation matrix $\bar{\mathbf{A}}^I$ simplifies all expressions.

We assume that matrix $\bar{\mathbf{A}}^R$ is positive definite (Fedorov, 1968; Auld, 1973; Helbig, 1994; Carcione, 2001). Similarly, we assume $\bar{\mathbf{A}}^I$ is positive definite or zero (Červený and Pšenčík, 2006, p. 1302). These assumptions are valid for both sign options of the exponential time factor. In the literature, both signs are used. Consequently, one must be careful when comparing the signs in equations given by different authors.

There are few studies of the Q-factor of plane waves propagating in homogeneous, viscoelastic anisotropic media. Among them are the important contributions by Carcione (1992, 1994, 2000, 2001, 2007) and Carcione and Cavallini (1993, 1995a). For plane SH-waves propagating in the plane of symmetry of an anisotropic medium, we refer to Krebes and Le (1994). For plane P-waves propagating in transversely isotropic (TI) and orthorhombic media, we refer to Zhu and Tsvankin (2006, 2007), who base their concept of attenuation exclusively on homogeneous plane waves. Their concept is not used in this paper. The expression for the Q-factor for homogeneous plane SH-waves propagating in homogeneous media of unrestricted viscoelasticity and anisotropy is found in Carcione (2001, 2007). The expression for Q, valid for homogeneous and inhomogeneous plane waves propagating in media of unrestricted anisotropy and viscoelasticity, is found in Červený and Pšenčík (2006). Using Carcione's (2001) definition of Q ($Q^{-1} = W_d/2U$) and the so-called mixed specification of the slowness vector, Červený and Pšenčík (2006) derive an exact formula in which Q is expressed in terms of the viscoelastic moduli and complex-valued slowness and polarization vectors of homogeneous or inhomogeneous waves. To obtain exact Q, one tone must solve an algebraic equation of the sixth degree with complex-valued coefficients. Červený and Pšenčík (2008) show that solving the algebraic equation of the sixth degree can be avoided if weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media are considered and the perturbation approach is applied. We denote the corresponding quality factor as $\hat{Q}$ and derive and discuss the relevant perturbation formula for it.

The first expression for the direction-dependent quality factor for generally anisotropic, weakly dissipative media, mathematically identical with $\hat{Q}$, is given by Gajewski and Pšenčík (1992). To derive $\hat{Q}$, Gajewski and Pšenčík (1992) use a quite different approach, based on modifying the asymptotic ray-series method. The inhomogeneous waves are not explicitly considered in the derivation. The same expression for $\hat{Q}$ is also derived by Červený (2001, equation 5.5.21), using the first-order perturbation equation for traveltime. Neither of these publications provides answers to two important questions: (1) How is the expression derived for $\hat{Q}$ related to the definition $Q^{-1} = W_d/E$? (2) Is the result also valid for inhomogeneous waves when inhomogeneous waves were not considered in the derivation?

We answer these two questions in this paper. We show that the expression for the quality factor, given by Gajewski and Pšenčík (1992) and Červený (2001), is consistent with the definition $Q^{-1} = W_d/E$. In addition, it is valid not only for homogeneous but also for weakly inhomogeneous waves. From formulas given by Gajewski and Pšenčík (1992), $\hat{Q}$ for anisotropic weakly dissipative media can also be expressed in terms of a complex-valued energy velocity (ray velocity) v_r: $\hat{Q}^{-1} = -2\,\mathrm{Im}(v_r)/\mathrm{Re}(v_r)$.

We investigate only the Q-factor of plane waves propagating in unbounded homogeneous, dissipative anisotropic media. Vavryčuk (2007a, 2007b) investigates the attenuation of waves generated by a point source in a similar medium. He defines the so-called ray-quality factor by the relation $Q^{\mathrm{ray}} = -\mathrm{Re}(v_r^2)/\mathrm{Im}(v_r^2)$, where v_r is the complex-valued energy velocity (Vavryčuk, 2007b, his equation 24). For weakly dissipative anisotropic media, this relation immediately yields the expression $\hat{Q}^{-1} = -2\,\mathrm{Im}(v_r)/\mathrm{Re}(v_r)$, given by Gajewski and Pšenčík (1992), valid locally even for heterogeneous media. A uniform treatment of the Q-factor in weakly dissipative anisotropic media, valid for both plane waves and waves generated by point sources, can be found in Červený et al. (2007).

In this paper, we speak of velocity isotropy or anisotropy when the referenced perfectly elastic medium, specified by the moduli a^R_{ijkl}, is isotropic or anisotropic. We speak of Q-factor isotropy or anisotropy when the Q-factor is or is not directionally dependent. For nonvanishing dissipation moduli a^I_{ijkl}, the velocity anisotropy generally implies Q-factor anisotropy. Only for a special choice of dissipation matrix $\bar{\mathbf{A}}^I$ related to $\bar{\mathbf{A}}^R$ by the relation $\bar{\mathbf{A}}^I = \bar{\mathbf{A}}^R/q$, where q is a real-valued positive constant, do we obtain strict Q-factor isotropy.

Our paper begins with a brief review of the theory of time-harmonic, homogeneous and inhomogeneous plane waves propagating in general viscoelastic anisotropic media. We use the mixed specification of slowness vector $\mathbf{p}$ (Červený and Pšenčík, 2005a), which reduces the problem of determining $\mathbf{p}$ to the solution of an algebraic equation of the sixth degree with complex-valued coefficients. Once the equation is solved, the exact expression for the Q-factor can be evaluated for all P, S1, and S2 homogeneous and inhomogeneous plane waves. For weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media, we substitute the numerical solution of the algebraic equation by an approximate but simple first-order perturbation formula. The expression for Q-factor then simplifies considerably. We show that the perturbation formula for $\hat{Q}$, which follows from Buchen's (1971) definition $Q^{-1} = W_d/E$, is related explicitly to the direction of the energy flux and is independent of the inhomogeneity of the considered plane wave. It is inversely proportional to the intrinsic attenuation factor $\mathcal{A}^{in}$; $\hat{Q}^{-1} = \mathcal{A}^{in}$, where $\mathcal{A}^{in}$ is a scalar quantity, characterizing intrinsic dissipative properties of the medium.

Finally, we study the relation of $\hat{Q}$ to attenuation coefficient α; we show that both are independent of the inhomogeneity of the plane wave under consideration only if they are considered along the ray direction. We illustrate the behavior of Q^{-1} and $\hat{Q}^{-1}$ on numerical examples for a model of a viscoelastic anisotropic medium. Appendix A contains explicit formulas for $\hat{Q}$ for weakly dissipative media of higher-symmetry anisotropy, both for arbitrarily strong anisotropy as well as for the weak-anisotropy approximation.

Throughout our paper, we use Cartesian coordinates x_i and time t. The lower- case italic indices $i, j, \ldots$ take the values 1,2,3; the Greek indices $\alpha, \beta, \ldots$ take the values 1,2,..., 6. The Einstein summation convention over repeated indices is used. For time-harmonic waves, we consider the exponential time factor $\exp(-\mathrm{i}\omega t)$, where ω is a fixed, real-valued, positive, circular frequency. The dots above letters denote partial derivatives with respect to time ($\ddot{u}_i = \partial^2 u_i/\partial t^2$), and the index following the comma in the subscript indicates the partial derivative with respect to the relevant Cartesian coordinate ($u_{i,j} = \partial u_i/\partial x_j$).

TIME-HARMONIC PLANE WAVES IN VISCOELASTIC ANISOTROPIC MEDIA

Time-harmonic plane waves propagating in an unbounded homogeneous viscoelastic anisotropic medium can be described by the expression

$$u_i(x_j, t) = aU_i \exp[-\mathrm{i}\omega(t - p_n x_n)], \tag{3}$$

where ω is a circular frequency; a is the scalar amplitude; and u_i, U_i, and p_i are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, the normalized polarization vector $\mathbf{U}$, ($\mathbf{U}\cdot\mathbf{U} = 1$), and the slowness vector $\mathbf{p}$, respectively. The values $\mathbf{U}$, $\mathbf{p}$, and a are independent of $\mathbf{x}$ and t. The vector $\mathbf{u}$ satisfies the equation of motion $\tau_{ij,j} = \rho\ddot{u}_i$, where ρ is density and τ_{ij} is the stress tensor, related to the infinitesimal strain tensor $e_{ij} = (u_{i,j} + u_{j,i})/2$ by the generalized Hooke's law $\tau_{ij} = c_{ijkl}e_{kl}$. Quantities c_{ijkl} are complex-valued viscoelastic moduli.

It follows from the equation of motion that equation 3 represents a plane wave if and only if U_1, U_2, and U_3 satisfy the system of three linear equations:

$$(\Gamma_{ik} - \delta_{ik})U_k = 0, \quad i = 1,2,3, \tag{4}$$

where Γ_{ik} is the generalized complex-valued Christoffel matrix, given by the relation

$$\Gamma_{ik} = a_{ijkl}p_j p_l, \tag{5}$$

and where δ_{ik} is the Kronecker symbol. System 4 can be solved for U_k only if

$$\det(\Gamma_{ik} - \delta_{ik}) = 0. \tag{6}$$

Equation 6 represents a constraint relation for complex-valued $\mathbf{p}$.

We express $\mathbf{p}$ using the so-called mixed specification, proposed by Červený (2004) (see also Červený and Pšenčík, 2005a):

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m}, \quad \mathbf{n}\cdot\mathbf{m} = 0. \tag{7}$$

The two real-valued, mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$ and the real-valued scalar quantity D uniquely specify the plane wave under consideration and can be chosen arbitrarily. Vector $\mathbf{n}$ specifies the direction of vector Re $\mathbf{p}$, and $\mathbf{m}$ is perpendicular to $\mathbf{n}$. Vectors $\mathbf{n}$ and $\mathbf{m}$ specify the propagation-attenuation plane. The quantity D is called the inhomogeneity parameter because it controls the inhomogeneity of the plane wave. For $D = 0$, the plane wave is homogeneous, for $D \neq 0$ it is inhomogeneous, and for $|D|$ small, it is weakly inhomogeneous.

The complex-valued quantity σ in equation 7 can be determined from the constraint relation 6, which, after inserting equation 7, takes the form (Červený and Pšenčík, 2005a)

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0. \quad (8)$$

Equation 8 is an algebraic equation of the sixth degree for σ with complex-valued coefficients. It has six complex-valued roots corresponding to P, S1, and S2 plane waves, propagating in the directions of **n** and $-\mathbf{n}$. Quantity σ is a function of a_{ijkl}, **n**, **m**, and D.

For certain simple cases, the algebraic equation of the sixth degree can be factorized to two equations: one of the fourth degree and the other of the second degree. Quantity σ of a plane wave propagating in an isotropic viscoelastic medium or of an SH-wave propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) medium is the solution of the quadratic equation resulting from factorizing equation 8. If we know σ, formula 7 for the slowness vector can be expressed in a closed analytical form.

It is often useful to express **p** as

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}, \quad (9)$$

where **P** and **A** are real-valued vectors, called the propagation and attenuation vectors, respectively. Vector **P** is perpendicular to the wavefront defined by equation $P_i x_i = \text{const}$. Both **P** and **A** are situated in the propagation-attenuation plane formed by **n** and **m**. It follows from equations 7 and 9 that **P** and **A** are given by the relations

$$\mathbf{P} = \mathbf{n}\,\mathrm{Re}\,\sigma, \quad \mathbf{A} = \mathbf{n}\,\mathrm{Im}\,\sigma + D\mathbf{m}. \quad (10)$$

The real-valued phase-velocity $\mathcal{C}$ in the direction of **n** and attenuation angle γ (the angle between **P** and **A**) are then expressed as follows:

$$\mathcal{C} = \frac{1}{|\mathbf{P}|} = \frac{1}{|\mathrm{Re}\,\sigma|}, \quad \cos\gamma = \frac{\varepsilon\,\mathrm{Im}\,\sigma}{[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2}}, \quad (11)$$

where $\varepsilon = \mathrm{Re}\,\sigma/|\mathrm{Re}\,\sigma| = \pm 1$. Once σ is found, we can determine **U** from equation 4 and from the normalization condition $\mathbf{U}\cdot\mathbf{U} = 1$.

We use D to specify the inhomogeneity of the wave and equation 7 to describe **p**. We call the inhomogeneous plane waves with small $|D|$ (not with small γ) weakly inhomogeneous plane waves. Červený and Pšenčík (2005a, 2005b) show that attenuation angle γ is not a suitable parameter for characterizing the inhomogeneity of plane waves, particularly in weakly dissipative media. One reason is that in weakly dissipative media, with $|a^I_{ijkl}|$ small, γ is very unstable. It can take any value between 0° and $\gamma_{\max}$, where $\gamma_{\max}$ depends on velocity anisotropy. It is usually close to 90° and may exceed it. For example, in a perfectly elastic isotropic medium, γ is always 90°. However, weakly inhomogeneous waves, including homogeneous waves ($\gamma = 0°$), can propagate in a weakly dissipative isotropic medium (arbitrarily close to a perfectly elastic isotropic medium). These observations follow from equation 11, considering $\mathrm{Im}\,\sigma$ small (weak dissipation).

Another reason that γ is not a suitable parameter for characterizing the inhomogeneity of plane waves is that so-called forbidden directions can be generated. In these directions, the square of the phase velocity $\mathcal{C}^2$ is negative (Krebes and Le, 1994; Carcione and Cavallini, 1995b; Carcione, 2007), which is, of course, physically unacceptable. Červený and Pšenčík (2005a, 2005b) explain this phenomenon in detail.

EXACT EXPRESSIONS FOR Q-FACTOR

We use the classical definition of Q-factor, given by Buchen (1971):

$$Q^{-1} = \frac{W_d}{E}. \quad (12)$$

If σ and **U** are known, we can compute the densities of the time-averaged energy-related quantities, e.g., the kinetic K, strain U, complete $E = K + U$ and dissipated W_d energies, the complex-valued Poynting vector **F** (also called the complex-valued energy flux), and the real-valued Poynting vector $\mathbf{S} = \mathrm{Re}\,\mathbf{F}$ (also called the energy flux).

The most important expression is for **F**. Its Cartesian components F_i for a plane wave as defined in equation 3 are given by the relation (Červený and Pšenčík, 2006, their equation 24)

$$F_i = c a_{ijkl} p_l U_j^* U_k, \quad (13)$$

where

$$c = \frac{1}{2}\rho\omega^2|a|\exp(-2\omega A_n x_n). \quad (14)$$

In equation 14, ρ is the density and a is the scalar amplitude. Consequently, c is a positive real-valued constant, and the time-averaged vector **S** has the form

$$S_i = c\,\mathrm{Re}(a_{ijkl} p_l U_j^* U_k). \quad (15)$$

In equation 12, we can use the relations

$$E = \mathbf{P}\cdot\mathbf{S}, \quad W_d = 2\mathbf{A}\cdot\mathbf{S}, \quad (16)$$

derived by Carcione and Cavallini (1993); see also Červený and Pšenčík (2006). In equations 16, **P** and **A** are given by equation 10 and **S** by equation 15. Inserting equations 16 into equation 12, we obtain the expression for Q:

$$Q^{-1} = \frac{2\mathbf{A}\cdot\mathbf{S}}{\mathbf{P}\cdot\mathbf{S}}. \quad (17)$$

Equation 17 for Q, with equation 10 for **P** and **A** and equation 15 for **S**, is exact. It is valid for homogeneous and inhomogeneous plane waves propagating in isotropic or anisotropic media, either perfectly elastic or viscoelastic. No approximation is used in its derivation, and no restrictions are imposed on the anisotropy or viscoelasticity of the medium and on the inhomogeneity of the plane wave. It is obvious from equation 17 that Q does not depend on c, specified in equation 14.

Figure 1 illustrates the most important quantities related to propagating an inhomogeneous plane wave in a homogeneous viscoelastic anisotropic medium.

THE Q-FACTOR FOR WEAKLY INHOMOGENEOUS PLANE WAVES IN ANISOTROPIC, WEAKLY DISSIPATIVE MEDIA

In this section, we use the first-order perturbation method to derive an approximate expression for the quality factor of $\hat{Q}$ of a weakly inhomogeneous plane wave propagating in an anisotropic, weakly dissipative medium ($|a^I_{ijkl}|$ small). In practical problems of seismology and seismic exploration, the dissipation is usually small and the inhomogeneity of the considered waves is also weak. In such a case, the numerical solution of the algebraic equation 8 of the sixth degree can be substituted by its approximate explicit solution, and the quality factor can be expressed in a simple analytical form.

As usual in perturbation methods, we use the word "weakly" only as a qualitative description in the terms weakly inhomogeneous plane waves and weakly dissipative media. The accuracy of perturbation equations increases with decreasing inhomogeneity of the studied plane waves and with decreasing dissipation of the medium. Quantitative estimation of the error of the perturbation equations would not be simple. The simplest estimation is to compare the results of exact and perturbation equations for several test situations, as we do in the section on numerical examples. This is one of the reasons for presenting exact equations.

The problem of weakly inhomogeneous plane waves propagating in weakly dissipative media is treated by Červený and Pšenčík (2008), who use the first-order perturbation method to solve it. As a reference (unperturbed) case, a homogeneous plane wave with a real-valued slowness vector ($D = 0$) propagating in a perfectly elastic medium ($a^I_{ijkl} = 0$) is considered. The quantities corresponding to this reference case are denoted below by upper indices 0. Therefore, $a^0_{ijkl} = a^R_{ijkl}$.

The solutions for the reference case are well known. For σ^0, $\mathbf{P}^0$, and $\mathbf{A}^0$, we have

$$\sigma^0 = \frac{1}{\mathcal{C}^0}, \quad \mathbf{P}^0 = \frac{\mathbf{n}}{\mathcal{C}^0}, \quad \mathbf{A}^0 = \mathbf{0}. \tag{18}$$

Symbol $\mathcal{C}^0$ denotes the phase velocity in the direction of $\mathbf{n}$ in the reference medium. The polarization vector $\mathbf{U}^0$ of the reference homogeneous wave can be determined from equation 4, specified for the perfectly elastic reference medium:

$$(\Gamma^0_{ik} - \delta_{ik})U^0_k = 0, \quad i = 1,2,3. \tag{19}$$

As in equation 4, we consider $\mathbf{U}^0$ to be normalized, $U^0_i U^0_i = 1$. The real-valued energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ of a homogeneous plane wave propagating in the perfectly elastic reference medium is given by the relation

$$\mathcal{U}^0_i = a^0_{ijkl} p^0_l U^0_j U^0_k. \tag{20}$$

Multiplying equation 19 by U^0_i, we obtain the relation $a^0_{ijkl} p^0_j p^0_l U^0_i U^0_k = 1$ and the relations

$$\mathcal{U}^0_i p^0_i = 1, \quad \mathcal{U}^0_i n_i = \mathcal{C}^0. \tag{21}$$

The inhomogeneity of the studied plane wave (specified by D) and the imaginary parts of the viscoelastic moduli (a^I_{ijkl}) are small perturbations. Unit vectors $\mathbf{n}$ and $\mathbf{m}$ remain fixed and are not perturbed. Under these assumptions, Červený and Pšenčík (2008) derive the following approximate expression for quantity σ:

$$\sigma = (\mathcal{C}^0)^{-1}\left[1 + \mathrm{i}\frac{\mathcal{A}^{\mathrm{in}}}{2} - \mathrm{i}D(\mathcal{U}^0 \cdot \mathbf{m})\right]. \tag{22}$$

We call quantity $\mathcal{A}^{\mathrm{in}}$ the intrinsic attenuation factor because it depends only on the intrinsic dissipation of the medium, not on D. It is given by the relation

$$\mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k. \tag{23}$$

For the perturbed propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$, we get from equation 10

$$\mathbf{P} = \frac{\mathbf{n}}{\mathcal{C}^0}, \quad \mathbf{A} = \frac{\mathbf{n}\mathcal{A}^{in}}{2\mathcal{C}^0} + D\left[\mathbf{m} - \mathbf{n}\frac{(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})}{\mathcal{C}^0}\right]. \tag{24}$$

Thus, the perturbed $\mathbf{P}$ is the same as the unperturbed, i.e., $\mathbf{P} = \mathbf{P}^0$. The perturbed $\mathbf{A}$, however, differs from the unperturbed significantly. The vector $\mathbf{A}^0$ is zero (equation 18), but $\mathbf{A}$ depends on dissipation moduli a^I_{ijkl} as well as on inhomogeneity parameter D.

Now we take the exact relation 17 for Q^{-1} and use it to derive the first-order perturbation relation for $\hat{Q}^{-1}$. Because $\mathbf{A}^0 = 0$, we can substitute $\mathbf{S}$ and $\mathbf{P}$ in equation 17 by their values $\mathbf{S}^0$ and $\mathbf{P}^0$ in the reference medium. Equation 15 yields

$$S^0_i = c\,\mathrm{Re}(a^0_{ijkl} p^0_l U^0_j U^0_k) = c\mathcal{U}^0_i, \tag{25}$$

where c is a real-valued positive constant given in equation 14 and $\boldsymbol{\mathcal{U}}^0$ is the energy-velocity vector in the reference medium (equation 20). Multiplying equation 25 successively by $\mathbf{A}$ and $\mathbf{P} = \mathbf{P}^0$, we obtain $A_j S^0_j = cA_j\mathcal{U}^0_j$ and $P^0_j S^0_j = cp^0_j\mathcal{U}^0_j = c$. We insert this into relation 17 and get the simple first-order perturbation formula:

$$\hat{Q}^{-1} = 2\mathbf{A} \cdot \boldsymbol{\mathcal{U}}^0. \tag{26}$$

The caret over Q emphasizes that $\hat{Q}$ is derived by the first-order perturbation method and is valid for weakly inhomogeneous plane waves propagating in weakly dissipative media only.

Inserting $\mathbf{A}$ from equation 24 into equation 26, we get the final expression for $\hat{Q}^{-1}$:

$$\hat{Q}^{-1} = \mathcal{A}^{\mathrm{in}} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k. \tag{27}$$

The term depending on D in the expression for $\mathbf{A}$ in equation 24 vanishes when multiplied by $\boldsymbol{\mathcal{U}}^0$. Thus, the $\hat{Q}$-factor in equation 27 does

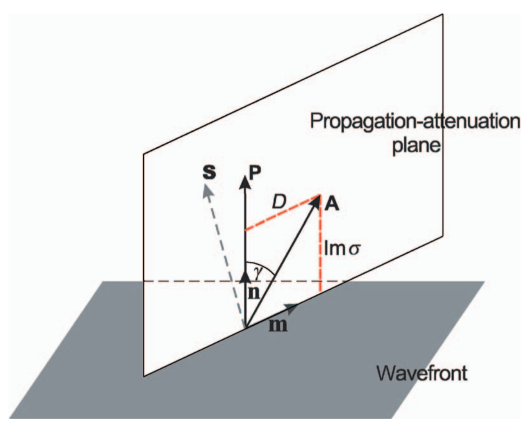

Figure 1. Diagram showing the most important quantities related to inhomogeneous plane-wave propagation in anisotropic dissipative media. Two real-valued, mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$ are used to define complex-valued slowness vector $\mathbf{p}$, given by equation 7. Vector $\mathbf{n}$ is perpendicular to the wavefront and specifies the direction of propagation-vector $\mathbf{P}$. In turn, $\mathbf{P}$ and the attenuation vector $\mathbf{A}$ define propagation attenuation plane. Vectors $\mathbf{P}$ and $\mathbf{A}$ make an attenuation angle γ. Projections of $\mathbf{A}$ into $\mathbf{n}$ and $\mathbf{m}$ are Im σ and the inhomogeneity parameter D, respectively. As indicated, the real-valued time-averaged Poynting vector $\mathbf{S}$, given by equation 15, may point out of the propagation-attenuation plane.

not depend on D, regardless of whether inhomogeneous plane waves (with $D \neq 0$) are considered.

We have shown thus far that for weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media, the exact Q-factor $Q = E/W_d$ reduces to the approximate form $\hat{Q}$ given by equation 27. It does not depend on D of the plane wave under consideration; it depends only on the intrinsic dissipation parameters of the medium. In the next section, we show that we should associate $\hat{Q}$ with the direction of unit vector $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Then $\hat{Q}$ is inversely proportional to attenuation coefficient $\alpha(\mathbf{t})$, which does not depend on D. The plane wave under consideration can be inhomogeneous, with attenuation vector $\mathbf{A}$ depending on D (see equation 24).

The equation for quality factor, mathematically identical to equation 27, is derived by Gajewski and Pšenčík (1992, their equation 7). They use modification of the asymptotic ray-series method, in which they assume that dissipation moduli a^I_{ijkl} are small quantities of the order of ω^{-1} for $\omega \to \infty$. The method is valid locally for high-frequency elementary waves propagating in smoothly heterogeneous, weakly dissipative, generally anisotropic media. The resulting direction-dependent quality factor corresponds to the attenuation along the reference ray direction. Consequently, it corresponds to the attenuation along energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ (not along propagation vector $\mathbf{P}^0$). Although Gajewski and Pšenčík (1992) consider neither Buchen's (1971) definition of Q nor inhomogeneity of waves, their equation 7 is fully consistent with our equation 27. Thus, it is also consistent with definition 12 of Buchen (1971), and it is valid even for weakly inhomogeneous waves. Also identical to equation 27 is Červený's (2001) equation 5.5.28, obtained using the first-order perturbation equation for traveltime.

The intrinsic attenuation factor $\mathcal{A}^{in}$ is equivalent to $\hat{Q}^{-1}$. Consequently, using classical terminology, it could also be called the loss factor. However, we prefer the term intrinsic attenuation factor to emphasize the independence of $\mathcal{A}^{in}$ on D and its strict definition by equation 23.

For some special cases, equation 27 for $\hat{Q}$ simplifies. For example, we can consider the 6×6 matrix $\bar{\mathbf{A}}$ of viscoelastic moduli, given by the relation

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I = \bar{\mathbf{A}}^R\left(1 - \frac{\mathrm{i}}{q}\right), \tag{28}$$

where q is an arbitrary real-valued positive constant. This means we can express any a^I_{ijkl} as $a^I_{ijkl} = a^R_{ijkl}/q$. Equation 27 then yields

$$\hat{Q}^{-1} = \frac{a^R_{ijkl}p^0_j p^0_l U^0_i U^0_k}{q} = \frac{1}{q}. \tag{29}$$

Equation 29 is the consequence of relations 20 and 21. Gajewski and Pšenčík (1992) obtain a similar result; we too can conclude that under condition 28, $\hat{Q}$ is direction independent. We can thus speak of Q-factor isotropy.

In anisotropic dissipative media, $\hat{Q}$ is generally direction dependent if condition 28 is not satisfied; $\hat{Q}$ is then direction dependent even if dissipation moduli a^I_{ijkl} have an isotropic structure:

$$a^I_{ijkl} = \left(\frac{\lambda^I}{\rho}\right)\delta_{ij}\delta_{kl} + \left(\frac{\mu^I}{\rho}\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{30}$$

where $-\lambda^I$ and $-\mu^I$ are the imaginary parts of Lamé's viscoelastic moduli. Inserting equation 30 into equation 27 yields

$$\hat{Q}^{-1} = \frac{\lambda^I + \mu^I}{\rho}(p^0_i U^0_i)(p^0_j U^0_j) + \frac{\mu^I}{\rho}(p^0_i p^0_i). \tag{31}$$

If the perfectly elastic reference medium is anisotropic, i.e., if we deal with velocity anisotropy, then $\hat{Q}$ is direction dependent (even if the matrix of dissipation moduli has an isotropic structure 30). In this case, we can speak of Q-factor anisotropy. For the dissipation moduli given in equation 30, $\hat{Q}$ becomes approximately direction independent in weakly anisotropic media with weak anisotropy parameters of the same order as the weak dissipation parameters (see equations A-7–A-9).

If the reference medium is isotropic, equation 31 yields the well-known results for isotropic viscoelastic media:

$$\hat{Q}^{-1} = \frac{(\lambda^I + 2\mu^I)}{(\lambda^R + 2\mu^R)} = -\frac{\mathrm{Im}(v_P^2)}{\mathrm{Re}(v_P^2)} \quad \text{for P-waves}$$

and

$$\hat{Q}^{-1} = \frac{\mu^I}{\mu^R} = -\frac{\mathrm{Im}(v_S^2)}{\mathrm{Re}(v_S^2)} \quad \text{for S waves,}$$

where v_P and v_S are the complex-valued P- and S-wave phase velocities and where λ^R, μ^R and $-\lambda^I$, $-\mu^I$ are real and imaginary parts of complex-valued Lamé viscoelastic moduli. Moreover, $\mathbf{n}$ and $\mathbf{t}$ coincide in the reference isotropic medium. Thus, equation 27 is fully consistent with the most commonly used definition of the Q-factor in isotropic dissipative media and represents its natural generalization for anisotropic dissipative media.

The specifications of formula 27 for anisotropic media of higher symmetry and for weakly anisotropic media are given in Appendix A.

RELATION OF $\hat{Q}$ TO ATTENUATION COEFFICIENT α

Attenuation coefficient α is another important parameter characterizing dissipation. In this section, we study its relation to $\hat{Q}$. We discuss α of an arbitrary time-harmonic homogeneous or inhomogeneous plane wave defined in equation 3, specified by $\mathbf{n}$, $\mathbf{m}$, and D along an arbitrarily oriented straight-line profile in a homogeneous, viscoelastic, anisotropic medium. Using equation 9 in formula 3 for the plane wave, we can alter the formula to the following form:

$$u_i(x_j,t) = aU_i \exp[-\mathrm{i}\omega(t - P_n x_n)]\exp[-\omega A_n x_n]. \tag{32}$$

We now introduce the plane-wave dissipation filter $\mathcal{D}(\omega)$ by the relation

$$\mathcal{D}(\omega) = \exp[-\omega A_n x_n]. \tag{33}$$

Equation 32 can then be expressed as

$$u_i(x_j,t) = aU_i\mathcal{D}(\omega)\exp[-\mathrm{i}\omega(t - P_n x_n)]. \tag{34}$$

The physical meaning of the plane-wave dissipation filter is obvious. It describes the variation of amplitudes along an arbitrarily chosen straight profile. We specify the profile under consideration by the unit real-valued vector $\mathbf{l}$. We further introduce distance $s - s_0$, measured along profile $\mathbf{l}$ with an arbitrarily selected initial point s_0 on the profile so that $s - s_0$ is positive in the direction of $\mathbf{l}$. The plane-wave dissipation filter 33 along profile $\mathbf{l}$ is then given by

$$\mathcal{D}(\omega) = \exp[-\omega(s - s_0)\mathbf{A}\cdot\mathbf{l}] = \exp[-\alpha(\mathbf{l})(s - s_0)], \quad (35)$$

where we introduce the quantity $\alpha(\mathbf{l})$:

$$\alpha(\mathbf{l}) = \omega\mathbf{A}\cdot\mathbf{l}. \quad (36)$$

We call $\alpha(\mathbf{l})$ the attenuation coefficient (or absorption coefficient) of the studied homogeneous or inhomogeneous plane wave along profile $\mathbf{l}$. Attenuation coefficient $\alpha(\mathbf{l})$ is real valued, and its dimension is $[\text{length}]^{-1}$. For certain directions of $\mathbf{l}$, the attenuation coefficient may be negative and thus may represent the exponential growth of amplitude (for example, if profile $\mathbf{l}$ is oriented against the direction of propagation).

Inserting $\mathbf{A}$ from equation 10 into equation 36, we obtain a formally simple expression for $\alpha(\mathbf{l})$ along profile $\mathbf{l}$:

$$\alpha(\mathbf{l}) = \omega[(\mathbf{n}\cdot\mathbf{l})\mathrm{Im}\,\sigma + (\mathbf{m}\cdot\mathbf{l})D]. \quad (37)$$

This expression for $\alpha(\mathbf{l})$ is exact and may be used for any homogeneous or inhomogeneous, plane wave (P, S1, S2) propagating in an arbitrary, perfectly elastic or viscoelastic isotropic or anisotropic medium along any profile $\mathbf{l}$. The plane wave is specified by $\mathbf{n}$, $\mathbf{m}$, and D. To determine $\alpha(\mathbf{l})$ exactly using equation 37, we must solve algebraic equation 8 of the sixth degree in σ. For viscoelastic isotropic media, the algebraic equation of the sixth degree factorizes into quadratic algebraic equations for plane P- and S-waves.

For weakly inhomogeneous plane waves propagating in weakly dissipative media, we can simplify equation 37 by inserting the first-order perturbation formula 24 for attenuation vector $\mathbf{A}$:

$$\alpha(\mathbf{l}) = \left(\frac{\omega}{2\mathcal{C}^0}\right)(\mathcal{A}^{in}(\mathbf{n}\cdot\mathbf{l}) + 2D[\mathcal{C}^0(\mathbf{m}\cdot\mathbf{l}) - (\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m})]). \quad (38)$$

In equation 38, we use $\mathcal{A}^{in}$ given by equation 23. Expression 38 is approximate, but otherwise it is quite general. We assume, of course, that dissipation moduli a^I_{ijkl} and inhomogeneity parameter D are small. The effects of a^I_{ijkl} and D are coupled in equation 38 even for weakly inhomogeneous waves propagating in weakly dissipative media.

In seismological structural studies, in seismic exploration, or in nondestructive testing of materials by ultrasonic techniques, researchers primarily are interested in intrinsic dissipation, caused exclusively by a^I_{ijkl}. The attenuation coefficients, related fully or partly to D, are usually of no interest because they are not related to the properties of the tested rocks only but also to the properties of the wave used for testing. Therefore, a very important question arises: Does a profile $\mathbf{l}$ exist, along which attenuation coefficient $\alpha(\mathbf{l})$ does not depend on D?

An exact answer requires studying equation 37 numerically because Im σ in equation 37 also depends on D. However, for weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media, we can use equation 38 and obtain the answer immediately: attenuation coefficient $\alpha(\mathbf{l})$, given by equation 38, does not depend on D if

$$\mathcal{C}^0(\mathbf{m}\cdot\mathbf{l}) - (\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m}) = 0. \quad (39)$$

This equation is satisfied only if $\mathbf{l}$ is taken parallel to $\boldsymbol{\mathcal{U}}^0$:

$$\mathbf{l} \parallel \boldsymbol{\mathcal{U}}^0. \quad (40)$$

Thus, α does not depend on D if $\mathbf{l}$ is parallel to the unperturbed energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ (the ray direction in the reference medium). Hereinafter, we consider $\alpha(\mathbf{t})$ along profile $\mathbf{t}$, where $\mathbf{t}$ is the real-valued unit vector along $\boldsymbol{\mathcal{U}}^0$, $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Consequently, for profile $\mathbf{l}$ taken along $\mathbf{t}$, attenuation coefficient $\alpha(\mathbf{t})$ is given by the simple relation

$$\alpha(\mathbf{t}) = \frac{\omega}{(2\hat{Q}\mathcal{U}^0)}, \quad (41)$$

where $\hat{Q}$ is the direction-dependent quality factor given by equation 27.

Equation 41 shows that the attenuation coefficient measured along the direction of $\boldsymbol{\mathcal{U}}^0$ depends only on intrinsic attenuation parameters a^I_{ijkl} and does not depend on D of the studied plane wave. Attenuation coefficient $\alpha(\mathbf{t})$ is inversely proportional to $\hat{Q}$. We obtain this result using the first-order perturbation method. Consequently, the result is approximate and valid only for weakly inhomogeneous plane waves propagating in weakly dissipative media. A similar conclusion is drawn by Deschamps and Assouline (2000) in their numerical study of plane waves in orthorhombic anisotropic media.

Using equation 41, we can define $\hat{Q}$ in terms of $\alpha(\mathbf{t})$ along profile $\mathbf{l} = \mathbf{t}$:

$$\hat{Q}^{-1} = \frac{\alpha(\mathbf{t})\mathcal{U}^0}{\pi f}, \quad (42)$$

where f is the frequency (in hertz). This relation for $\hat{Q}$ is formally the same as the relation known for isotropic viscoelastic media, namely, $Q^{-1} = \alpha V/\pi f$, where V is the velocity of the relevant plane P- or S-wave (Johnston and Toksöz, 1981, their equation 36). In homogeneous, isotropic viscoelastic media, α, V, and Q are direction independent. In homogeneous, anisotropic viscoelastic media, $\alpha(\mathbf{t})$, $\mathcal{U}^0$, and $\hat{Q}$ depend on the direction of the real-valued vector $\mathbf{t}$. In this way, equation 42 represents a generalization of the relation $Q^{-1} = \alpha V/\pi f$ for anisotropic viscoelastic media.

To calculate $\hat{Q}$ corresponding to the direction of $\mathbf{t}$ using equation 27, one must know $\boldsymbol{\mathcal{U}}^0$ and the corresponding slowness vector $\mathbf{p}^0$, both in the reference medium. The real-valued vectors $\mathbf{p}^0$ and $\boldsymbol{\mathcal{U}}^0$ are mutually related (see equation 20). We can proceed in two ways.

The first possibility is to specify $\mathbf{p}^0$ and to calculate the corresponding $\boldsymbol{\mathcal{U}}^0$ from equation 20. This straightforward procedure is extremely simple. The S-wave singularities and the multivaluedness of S-wave surfaces cause no problems. It is typical for ray tracing in heterogeneous, anisotropic, weakly dissipative media (Gajewski and Pšenčík, 1992). A certain disadvantage is that $\boldsymbol{\mathcal{U}}^0$ and thus $\mathbf{t}$ are not known in advance but must be computed. Consequently, we cannot choose the profile of measurements in advance. The situation simplifies considerably for computations in any plane of symmetry.

The second possibility consists of directly computing $\hat{Q}$ for a given $\mathbf{t}$. In this case, $\mathbf{p}^0$ appearing in equation 27 is not known and should be calculated from a given $\boldsymbol{\mathcal{U}}^0$. This procedure is considerably more complicated than the previous one. It is very sensitive to S-wave singular regions and to the multivaluedness of the wave surfaces when several values of $\mathbf{p}^0$ can be obtained for one given $\boldsymbol{\mathcal{U}}^0$.

The procedure resembles the two-point ray-tracing procedure. To solve the problem, $\mathbf{p}^0$ is sought iteratively so that it yields the given $\boldsymbol{\mathcal{U}}^0$. Alternatively, the method based on the numerical solution of a system of polynomial equations, proposed by Vavryčuk (2006), can be used.

If we use profile $\mathbf{l}$ along energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ in equations 35 and 41, the final expression for the dissipation filter in weakly dissipative anisotropic media is as follows:

$$\mathcal{D}(\omega) = \exp\left[-\frac{\omega(s - s_0)}{2\hat{Q}\mathcal{U}^0}\right], \tag{43}$$

where $\hat{Q}$ is given by equation 27 and $\mathcal{U}^0 = |\boldsymbol{\mathcal{U}}^0|$ is given by equation 20. For weakly inhomogeneous plane waves, $\alpha(\mathbf{t})$, $\hat{Q}$, and dissipation filter $\mathcal{D}(\omega)$ do not depend on D of the wave. Equations 41 and 43 also remain valid for isotropic media, where $\mathcal{U}^0$ and $\hat{Q}$ are direction independent.

Equation 38 also can be used to investigate attenuation coefficients $\alpha(\mathbf{l})$ in directions $\mathbf{l}$ other than the direction of energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. In all these directions, however, $\alpha(\mathbf{l})$ depends on the inhomogeneity of the considered plane wave. For example, consider the case of the profile in the direction of vector $\mathbf{n}$, perpendicular to the wavefront in the reference medium and oriented in the direction of wave propagation. Equation 38 then yields

$$\alpha(\mathbf{n}) = \left(\frac{\omega}{2\mathcal{C}^0}\right)[\mathcal{A}^{in} - 2D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})]. \tag{44}$$

In anisotropic media, $\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m}$ is generally different from zero so that attenuation coefficient $\alpha(\mathbf{n})$ is practically always influenced by D of the considered wave. For certain $\mathbf{n}$, attenuation coefficient $\alpha(\mathbf{n})$ may even be negative (Červený and Pšenčík, 2005b). Consequently, profile $\mathbf{n}$ (perpendicular to the wavefront in the reference medium) is unsuitable for measuring attenuation coefficient and Q-factor in inhomogeneous anisotropic media.

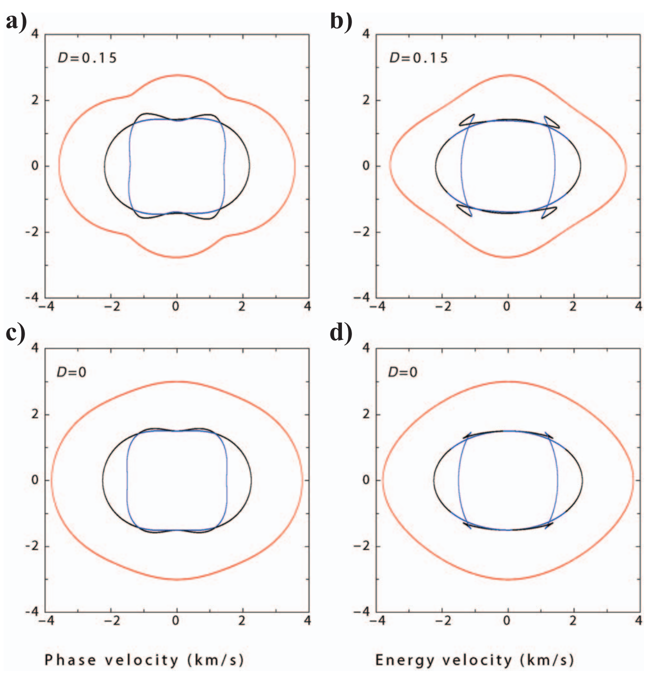

Figure 2. Polar diagrams of (a, c) phase velocity $\mathcal{C}$ and (b, d) energy velocity $\mathcal{U}$ in a plane of symmetry of the medium defined in equations 46 and 47 for $D = 0$ and 0.15 s/km: red is the fastest wave (P-wave), black is the S-wave with intermediate phase velocity, and blue is the slower S-wave. The velocity diagrams of the SH-wave are egg shaped. The SH-wave is faster than the SV-wave in the directions around the horizontal and, for $D = 0.15$ s/km, very close to the vertical.

NUMERICAL EXAMPLES

In this section, we illustrate the behavior of $\hat{Q}$ in a model of sedimentary rock and estimate the accuracy of the perturbation formula for $\hat{Q}^{-1} = \mathcal{A}^{in}$ by comparing its results with the exact results for Q^{-1}. The model is a modification of the one used by Zhu and Tsvankin (2006) and Vavryčuk (2007b). The 6×6 matrix $\bar{\mathbf{A}}$ of complex-valued, density-normalized viscoelastic moduli $\bar{A}_{\alpha\beta}$ in Voigt notation [measured in $(\mathrm{km/s})^2$] is as follows:

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I, \tag{45}$$

where

$$\bar{\mathbf{A}}^R = \begin{pmatrix} 14.40 & 4.40 & 4.50 & 0 & 0 & 0 \\ & 14.40 & 4.50 & 0 & 0 & 0 \\ & & 9.00 & 0 & 0 & 0 \\ & & & 2.25 & 0 & 0 \\ & & & & 2.25 & 0 \\ & & & & & 5.00 \end{pmatrix} \tag{46}$$

and

$$\bar{\mathbf{A}}^I = \begin{pmatrix} 0.48 & 0.15 & 0.30 & 0 & 0 & 0 \\ & 0.48 & 0.30 & 0 & 0 & 0 \\ & & 0.45 & 0 & 0 & 0 \\ & & & 0.15 & 0 & 0 \\ & & & & 0.15 & 0 \\ & & & & & 0.66 \end{pmatrix}. \tag{47}$$

Both matrices $\bar{\mathbf{A}}^R$ and $\bar{\mathbf{A}}^I$ are positive definite. Matrix $\bar{\mathbf{A}}^R$ corresponds to an anisotropic medium of hexagonal symmetry with a vertical axis of symmetry.

Although the formulas derived in previous sections can be used quite generally, for simplicity we concentrate on a symmetry plane of the above medium. We can choose it so that it is situated in the (x_1,x_3)-plane. In this case, the components of $\mathbf{n}$ (unit vector parallel to the propagation vector) and $\mathbf{t}$ (unit vector parallel to the energy-velocity vector) can be expressed as

$$n_1 = \sin i_{ph}, \quad n_2 = 0, \quad n_3 = \cos i_{ph},$$
$$t_1 = \sin i_{en}, \quad t_2 = 0, \quad t_3 = \cos i_{en}. \tag{48}$$

We call angle i_{ph} the phase angle and i_{en} the ray angle.

The following figures show polar diagrams parameterized by phase and ray angles, with $i = 0^\circ$ upward and $i = 90^\circ$ to the right. The polar diagrams in Figures 2–5 are plotted in the form used by Carcione (1992).

Figure 2 shows the polar diagrams of the exact phase velocity $\mathcal{C}$ (left column) and of the exact energy velocity $\mathcal{U}$ (right column) in the plane of symmetry of the medium defined in equations 46 and 47.

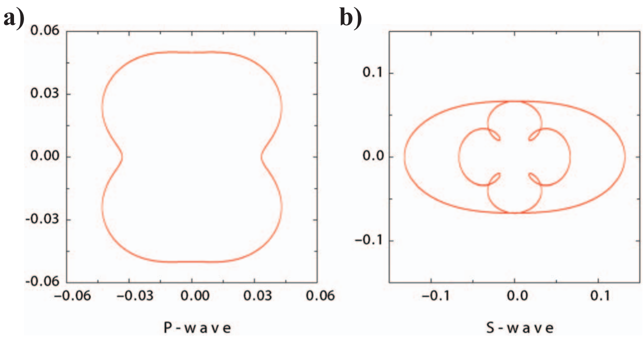

Figure 3. Polar diagrams of the inverse value of $\hat{Q}$ (dimensionless; $\hat{Q}^{-1} = \mathcal{A}^{in}$) calculated from equation 27 as a function of the ray angle. The (a) P-wave and (b) S-wave diagrams are in a plane of symmetry of the medium defined in equations 46 and 47. Note the inner smooth loops on the curve corresponding to the SV-wave. The loops are situated at the same ray angles as the triplications on the SV-wave energy-velocity curves; see Figure 2. The loops correspond to the lowest attenuation.

The phase-velocity diagrams are parameterized by the phase angle; the energy-velocity diagrams are parameterized by the ray angle. Red denotes the fastest wave (P-wave). Black corresponds to the S-wave with higher phase velocity, and blue corresponds to the slower S-wave. We see that different S-waves (SH or SV) are faster in different directions. From the polarization diagrams, we can identify the egg-shaped curves corresponding to the SH-waves.

The velocity diagrams are shown for two values of D: for $D = 0$ s/km (homogeneous wave) and for $D = 0.15$ s/km. We later prove that $D = 0.15$ s/km corresponds to rather strongly inhomogeneous waves. Because of strong S-wave velocity anisotropy, loops are formed on the SV-wave energy-velocity diagrams, even for $D = 0$. The size of the loops increases with increasing inhomogeneity of the wave. Figure 2 shows that the SV- and SH-wave phase-velocity curves intersect each other close to the vertical axis and close to phase angles $i_{ph} = \pi/4, 3\pi/4, \ldots$. They also show the mutual attraction of P- and SV-wave phase-velocity curves for increasing D. These interesting phenomena are described by Červený and Pšenčík

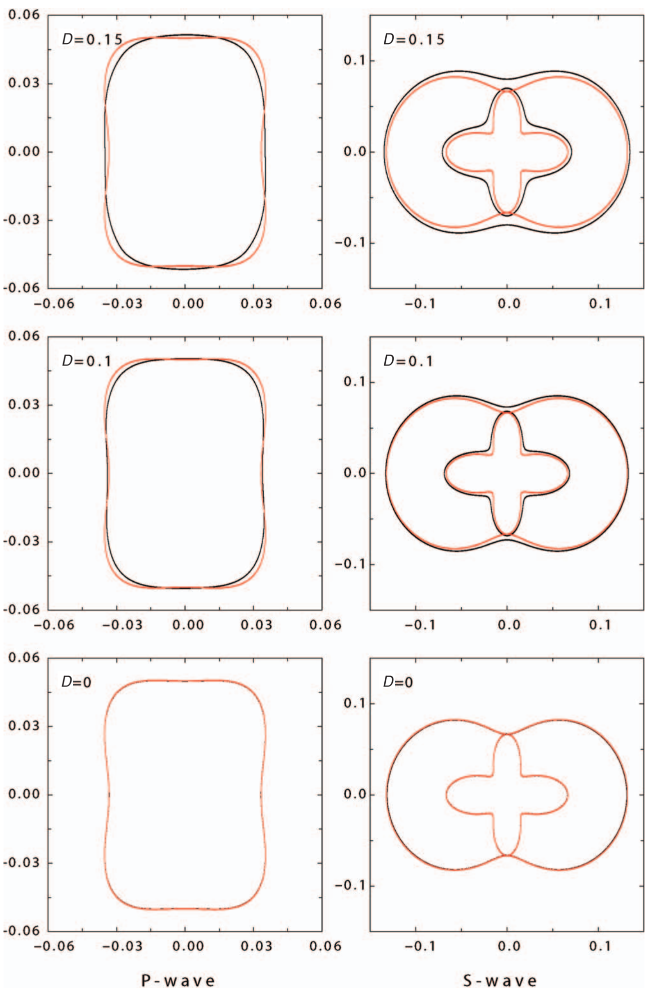

Figure 4. Comparison of polar diagrams of $\hat{Q}^{-1}(\hat{Q}^{-1} = \mathcal{A}^{in})$, calculated from equation 27 (red) with polar diagrams of exact Q^{-1} calculated from equation 17 (black) as a function of the phase angle. The values $\hat{Q}$ and Q are dimensionless. The P-wave (left) and S-wave diagrams (right) are in a plane of symmetry of the medium defined in equations 46 and 47 for inhomogeneity parameters $D = 0$, 0.1 and 0.15 s/km. The value of $\hat{Q}^{-1}$ (red) does not depend on inhomogeneity parameter D, but attenuation coefficient α does; see equation 44.

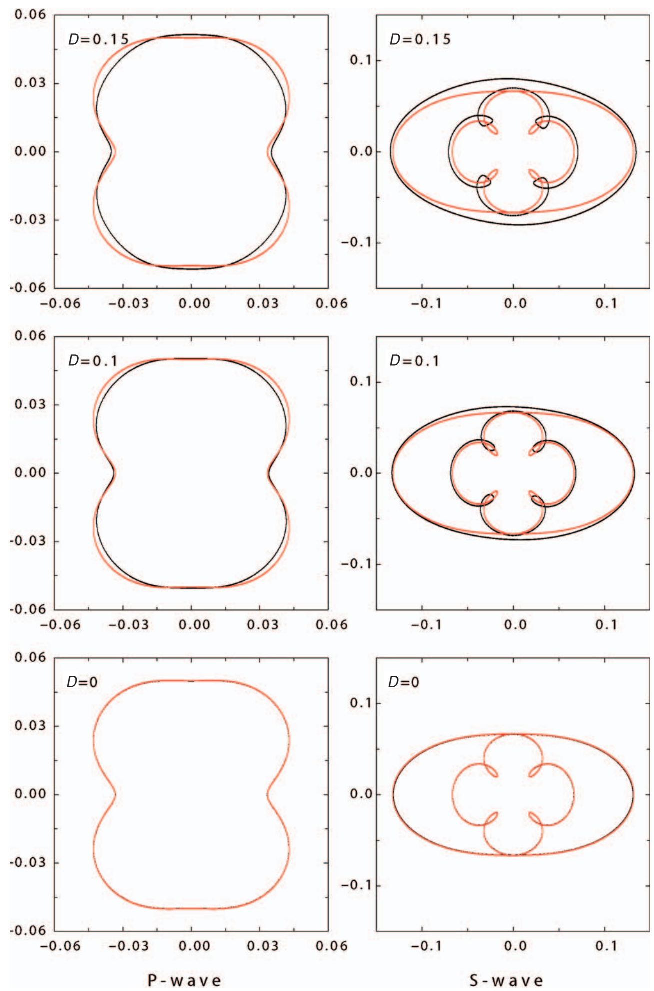

Figure 5. Comparison of polar diagrams of $\hat{Q}^{-1}(\hat{Q}^{-1} = \mathcal{A}^{in})$, calculated from equation 27 (red) with exact polar diagrams of Q^{-1} calculated from equation 17 (black) as a function of the ray angle. The values $\hat{Q}$ and Q are dimensionless. The P-wave (left) and S-wave diagrams (right) are in a plane of symmetry of the medium defined in equations 46 and 47 for inhomogeneity parameters $D = 0$, 0.1, and 0.15 s/km. Neither $\hat{Q}^{-1}$ (red) nor α depends on inhomogeneity parameter D; see equation 41.

(2005b). The analogous plots of the phase and energy velocities for a perfectly elastic reference medium $\bar{\mathbf{A}}^R$ from equation 46 are effectively identical with the plots in Figure 2.

In Figure 3, we show the polar diagrams of $\hat{Q}^{-1}$ calculated from formula 27 as a function of the ray angle (specified by the energy-velocity vector in the reference medium). The simpler plot (a) corresponds to the P-wave; the more complicated plots (b) correspond to both S-waves. The egg-shaped curve corresponds to the SH-wave, and the curve with internal loops corresponds to the SV-wave. Because of low P-wave attenuation anisotropy, the behavior of $\hat{Q}^{-1}$ for the P-wave is, in comparison with the SV-wave, rather simple. The strongest attenuation of P-wave can be observed in directions close to diagonals; in the SH-wave, in directions close to the horizontal; in the SV-wave, in directions close to both the vertical and the horizontal. The latter wave displays the strongest directivity of all three waves.

The smooth inner loops on the curve in the S-wave diagram, corresponding to the SV-wave, are the most remarkable phenomenon in Figure 3. The loops are situated at exactly the same ray angles at which we observe the outer loops on the SV-wave energy-velocity curves in Figure 2. At these loops, the values of $\hat{Q}^{-1}$ attain minimum values. Consequently, the attenuation of SV-waves is minimal in the regions of the loops. This important phenomenon, which we observe in other models, certainly deserves more investigation.

The profile of measurement corresponds to the direction of energy-velocity vector in the reference medium $\mathbf{l} = \mathbf{t}$, so both $\hat{Q}$ and α are independent of D of the wave under consideration and are related by equation 41. Consequently, they both are suitable for measuring intrinsic dissipative properties of anisotropic media.

In Figure 4, we compare the polar diagrams of $\hat{Q}^{-1} = \mathcal{A}^{in}$ (in red) with the polar diagrams of exact Q^{-1} (in black). The diagrams are again calculated in the plane of symmetry of the above medium. The polar diagrams are displayed as functions of phase angle i_{ph}, i.e., as functions of $\mathbf{n}$ parallel to the propagation vector. The left column shows the plots corresponding to the P-wave; the right column, to the S-waves. In both columns, plots are shown for inhomogeneity parameters $D = 0, 0.1$, and 0.15 s/km. The factor $\hat{Q}$ (calculated using equation 27) is independent of D, but the exact values of Q (calculated using equation 17) depend on D. Only for $D = 0$ s/km are $\hat{Q}$ and

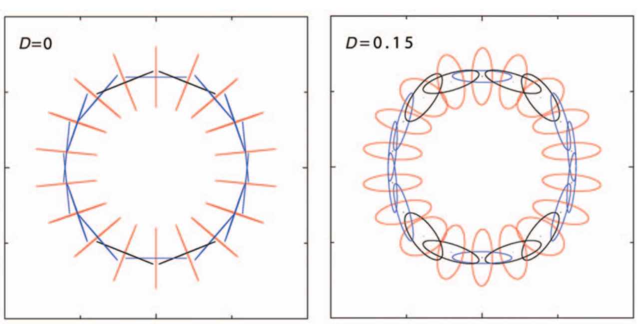

Figure 6. Particle motion polar diagrams in the plane of symmetry of the medium defined in equations 46 and 47 for $D = 0$ and 0.15 s/km. The colors are the same as in Figure 2. The SH-wave is polarized perpendicular to the plane of the plots. Note the effective linearity and strong ellipticity of the particle motion of the P-wave and SV-waves for $D = 0$ and 0.15 s/km, respectively. The ellipticity is caused primarily by the inhomogeneity of the considered waves. The value of $D = 0.15$ s/km thus corresponds to rather strongly inhomogeneous waves.

Q the same. The difference between them increases with increasing D. Because $D = 0.15$ s/km corresponds to a rather strong inhomogeneity of the wave under consideration (see Figure 6), we can conclude that $\hat{Q}^{-1}$ is a good approximation of the exact Q^{-1} for weakly inhomogeneous waves. The red curves, corresponding to $\hat{Q}^{-1}$, are plotted over the black curves, so that the red dominates with an exact fit. Consequently, it would be possible to use $\hat{Q}$ as a good approximation of Q in the direction of $\mathbf{n}$, $Q(\mathbf{n})$. The problem, however, is that $\alpha(\mathbf{n})$ along the profile in the direction of $\mathbf{n}$ is not intrinsic and depends on D (see equation 44). Consequently, profile $\mathbf{l} = \mathbf{n}$ is unsuitable for measuring the intrinsic dissipative properties of rocks in anisotropic media.

In Figure 5, we show the same comparison as Figure 4, the difference being the use of the ray angle i_{en} in Figure 5 instead of the phase angle i_{ph} in Figure 4. Consequently, both $\hat{Q}$ and α are independent of D and are suitable for measuring the intrinsic dissipative properties of anisotropic media. The system of plots and the use of colors is the same as in Figure 4. There is a remarkable difference of behavior of S-waves between Figures 4 and 5. The most remarkable phenomenon is the inner smooth loops on the SV-wave curves. Their position is the same for the exact and perturbation results. The width of the exactly calculated loops increases slightly with increasing inhomogeneity of the wave, whereas the perturbation result does not depend on the inhomogeneity of the wave.

To estimate the strength of the inhomogeneity of the studied waves, we present the particle motion diagrams for $D = 0$ (homogeneous wave) and for $D = 0.15$ s/km in the plane of symmetry of the above medium in Figure 6. The colors are the same as in Figure 2. In Figure 6, for $D = 0$ s/km, the polarization of all waves is effectively linear. The SH-wave is polarized linearly in both plots, perpendicular to the planes of the plots. In contrast to the homogeneous waves, the polarization of the inhomogeneous P- and SV-waves for $D = 0.15$ s/km is strongly elliptical, caused primarily by the inhomogeneity of the waves. This indicates that $D = 0.15$ s/km corresponds to the rather strong inhomogeneity of the wave. Good fit of $\hat{Q}$ and Q in Figure 5 thus implies that the applicability of perturbation formula 27 is rather broad. It may yield sufficiently accurate results even for waves of relatively strong inhomogeneity.

The above tests seem to indicate that equation 27 can be applied to inhomogeneous waves of considerable strength, propagating in anisotropic media of arbitrary symmetry and strength and of rather strong Q-factor anisotropy.

CONCLUSIONS

We have derived the simple perturbation formula 27 for $\hat{Q}$. It has several interesting and important properties.

First, expression 27 for $\hat{Q}$ is valid for homogeneous and weakly inhomogeneous time-harmonic P, S1, and S2 plane waves propagating in homogeneous, isotropic, or generally anisotropic weakly dissipative media. The factor $\hat{Q}$ is a positive scalar quantity, both in isotropic and anisotropic dissipative media. It does not depend on inhomogeneity parameter D of the considered plane wave, although its attenuation vector $\mathbf{A}$ might depend on D. Consequently, $\hat{Q}$ describes the intrinsic dissipative properties of the medium.

In anisotropic media, expression 27 for $\hat{Q}$ is fully consistent with the most common definitions of Q used in isotropic dissipative media and represents their natural generalization. It is generally direction dependent, the appropriate direction being specified by the direction of the energy-velocity vector (ray direction) and not by the direction of the propagation vector. This is because $\hat{Q}$ is inversely proportional to attenuation coefficient $\alpha(\mathbf{t})$, which does not depend on the inhomogeneity of the wave and corresponds to the profile parallel to the energy-velocity vector. The factor $\hat{Q}$ is direction dependent even if dissipation matrix $\bar{\mathbf{A}}^I$ in equation 45 has a structure that corresponds to an isotropic medium, i.e., Q-factor anisotropy is always related to velocity anisotropy. The Q-factor isotropy can exist strictly only in anisotropic media with property 28 or approximately in weakly anisotropic, weakly dissipative media.

Expression 27 for $\hat{Q}$ can be specified easily for an anisotropy of higher symmetry (orthorhombic, hexagonal) as well as for weak anisotropy. The computation of $\hat{Q}$ in the vicinity of S-wave singularities or in the case of the multivaluedness of wave surfaces does not cause any difficulties, as shown by the examples presented. The examples display certain interesting phenomena, especially in the regions of triplication of the S-wave wavefront, not known from previous studies.

Expression 27 for $\hat{Q}$ remains locally valid even for high-frequency elementary waves generated by point sources and propagating in heterogeneous, isotropic, or anisotropic weakly dissipative media.

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APPENDIX A

$\hat{Q}$ FOR HIGHER-ANISOTROPY SYMMETRIES AND WEAK ANISOTROPY

Equation 27 for $\hat{Q}^{-1}$ holds for all three waves — P, S1, and S2 — propagating in weakly dissipative anisotropic media of arbitrary symmetry and strength. Equations for the individual waves can be obtained from equation 27 by specifying the appropriate polarization vector $\mathbf{U}^0$ in it. Although it is expressed in terms of the unit vector $\mathbf{n}$, perpendicular to the wavefront in the perfectly elastic reference medium, $\hat{Q}$ in equation 27 corresponds to the direction of reference ray $\mathbf{t}$, $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Consequently, equation 27 must always be supplemented by equation 20, which provides the relation for determining $\mathbf{t}$ from $\mathbf{n}$. Equation 27 can be used without problems even in the case of wavefront triplication.

For models with higher material symmetry, it is more suitable to express dissipation moduli a^I_{ijkl} in Voigt notation $\bar{A}^I_{\alpha\beta}$. Here, $\bar{A}^I_{\alpha\beta}$ are elements of the 6×6 real-valued, symmetric, positive definite dissipation matrix $\bar{\mathbf{A}}^I$. The material symmetry of $\bar{\mathbf{A}}^I$ may, in general, differ from the material symmetry of $\bar{\mathbf{A}}^R$. For simplicity, we consider the material symmetry of $\bar{\mathbf{A}}^I$ to be equal to or higher than the symmetry of $\bar{\mathbf{A}}^R$.

Let us consider orthorhombic symmetry with its axes of symmetry coinciding with the axes of the Cartesian coordinate system, in which the x_1- and x_2-axes are horizontal and the x_3-axis is vertical, positive downward. The coordinate system is right-handed. In this case, dissipation matrix $\bar{\mathbf{A}}^I$ (similarly to matrix $\bar{\mathbf{A}}^R$) is specified by nine nonzero dissipation moduli: $\bar{A}^I_{11}$, $\bar{A}^I_{22}$, $\bar{A}^I_{33}$, $\bar{A}^I_{44}$, $\bar{A}^I_{55}$, $\bar{A}^I_{66}$, $\bar{A}^I_{12}$, $\bar{A}^I_{13}$, and $\bar{A}^I_{23}$. We can then express equation 27 for $\hat{Q}^{-1}$ as follows:

$$\begin{aligned}\hat{Q}^{-1} &= \mathcal{A}^{in}\\ &= \bar{A}^I_{11}(p^0_1U^0_1)^2 + \bar{A}^I_{22}(p^0_2U^0_2)^2 + \bar{A}^I_{33}(p^0_3U^0_3)^2\\ &\quad + \bar{A}^I_{44}(p^0_2U^0_3 + p^0_3U^0_2)^2 + \bar{A}^I_{55}(p^0_1U^0_3 + p^0_3U^0_1)^2\\ &\quad + \bar{A}^I_{66}(p^0_1U^0_2 + p^0_2U^0_1)^2 + 2\bar{A}^I_{12}p^0_1p^0_2U^0_1U^0_2\\ &\quad + 2\bar{A}^I_{13}p^0_1p^0_3U^0_1U^0_3 + 2\bar{A}^I_{23}p^0_2p^0_3U^0_2U^0_3. \qquad \text{(A-1)}\end{aligned}$$

For transverse isotropy with a vertical axis of symmetry (VTI), we have $\bar{A}^I_{11} = \bar{A}^I_{22}$, $\bar{A}^I_{44} = \bar{A}^I_{55}$, $\bar{A}^I_{13} = \bar{A}^I_{23}$, and $\bar{A}^I_{12} = \bar{A}^I_{11} - 2\bar{A}^I_{66}$. For $\hat{Q}^{-1}$, equation A-1 then yields

$$\begin{aligned}\hat{Q}^{-1} &= \mathcal{A}^{in}\\ &= \bar{A}^I_{11}(p^0_1U^0_1 + p^0_2U^0_2)^2 + \bar{A}^I_{33}(p^0_3U^0_3)^2 + \bar{A}^I_{44}[(p^0_2U^0_3\\ &\quad + p^0_3U^0_2)^2 + (p^0_1U^0_3 + p^0_3U^0_1)^2] + \bar{A}^I_{66}(p^0_1U^0_2\\ &\quad - p^0_2U^0_1)^2 + 2\bar{A}^I_{13}p^0_3U^0_3(p^0_1U^0_1 + p^0_2U^0_2). \qquad \text{(A-2)}\end{aligned}$$

Equations A-1 and A-2 hold for velocity anisotropy of unlimited strength and for all three waves: P, S1, and S2. By using $\mathbf{U}^0$ of one of the waves in equation A-1 or A-2, $\hat{Q}$ can be specified for that wave.

Equation 27 simplifies considerably if weak velocity anisotropy is considered. In this case, we can substitute the polarization vectors by determinable unit vectors (Farra and Pšenčík, 2003). For example, let us consider a P-wave with $U^0_i \approx n_i$, where n_i is the unit vector perpendicular to the wavefront in the reference medium. Equation 27 is then given by the relation

$$\hat{Q}^{-1}_P = \mathcal{A}^{\mathrm{in}}_P = (\mathcal{C}^0_P)^{-2}a^I_{ijkl}n_jn_ln_in_k. \qquad \text{(A-3)}$$

In equation A-3, $\mathcal{C}^0_P$ is the first-order P-wave phase velocity in the direction of vector $\mathbf{n}$. For P-waves in a weakly orthorhombic medium, equation A-1 yields

$$\begin{aligned}\hat{Q}^{-1}_P &= \mathcal{A}^{in}_P\\ &= (\mathcal{C}^0_P)^{-2}[\bar{A}^I_{11}n^4_1 + \bar{A}^I_{22}n^4_2 + \bar{A}^I_{33}n^4_3\\ &\quad + 2(\bar{A}^I_{12} + 2\bar{A}^I_{66})n^2_1n^2_2 + 2(\bar{A}^I_{13} + 2\bar{A}^I_{55})n^2_1n^2_3\\ &\quad + 2(\bar{A}^I_{23} + 2\bar{A}^I_{44})n^2_2n^2_3]. \qquad \text{(A-4)}\end{aligned}$$

Similarly, for P-waves in a weakly VTI medium, equation A-2 yields

$$\hat{Q}_P^{-1} = \mathcal{A}_P^{\text{in}} = (\mathcal{C}_P^0)^{-2}[\bar{A}_{11}^I(1-n_3^2)^2 + \bar{A}_{33}^I n_3^4 + 2(\bar{A}_{13}^I + 2\bar{A}_{55}^I)n_3^2(1-n_3^2)^2]. \quad \text{(A-5)}$$

We can see that the velocity anisotropy, expressed by direction-dependent $\mathcal{C}_P^0$, always affects the Q-factor anisotropy expressed by direction-dependent $\hat{Q}_P$.

We can express the square of first-order P-wave phase velocity $\mathcal{C}_P^0$ as

$$(\mathcal{C}_P^0)^2 = \bar{\alpha}^2[1 + 2Y(n_i)], \quad \text{(A-6)}$$

where $Y(n_i)$ stands for the linear combination of weak-anisotropy parameters, with coefficients containing powers of n_i. By symbol $\bar{\alpha}$, we denote a formal parameter with the dimension of velocity, used to define dimensionless weak-anisotropy parameters. We use a bar over $\bar{\alpha}$ to distinguish it from attenuation coefficient α. We can choose the nonzero parameter $\bar{\alpha}$ arbitrarily. For the explicit expression for $Y(n_i)$ for P-waves and for more details, refer to Pšenčík and Farra (2005, their equation 12).

If the weak-anisotropy parameters are of the same order as the dissipation parameters, we can simplify equations A-4 and A-5 further. We substitute the phase velocity $\mathcal{C}_P^0$ by $\bar{\alpha}$ and then express equation A-4 as

$$\hat{Q}_P^{-1} = \mathcal{A}_P^{\text{in}} = (Q_P^{\text{iso}})^{-1}[1 + 2(\varepsilon_x^I n_1^4 + \varepsilon_y^I n_2^4 + \varepsilon_z^I n_3^4 + \delta_x^I n_1^2 n_3^2 + \delta_y^I n_2^2 n_3^2 + \delta_z^I n_1^2 n_2^2)]. \quad \text{(A-7)}$$

Equation A-5 becomes

$$\hat{Q}_P^{-1} = (Q_P^{\text{iso}})^{-1}(1 + 2[\varepsilon_x^I(1-n_3^2)^2 + \varepsilon_z^I n_3^4 + \delta_x^I(1-n_3^2)n_3^2]). \quad \text{(A-8)}$$

In this case, the velocity anisotropy does not affect Q-factor anisotropy.

We define the coefficients in equations A-7 and A-8 in a way similar to weak-anisotropy parameters (Pšenčík and Farra, 2005):

$$\epsilon_x^I = \frac{(\bar{A}_{11}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{2\bar{\alpha}^2},$$

$$\epsilon_y^I = \frac{(\bar{A}_{22}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{2\bar{\alpha}^2},$$

$$\epsilon_z^I = \frac{(\bar{A}_{33}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{2\bar{\alpha}^2},$$

$$\delta_x^I = \frac{(\bar{A}_{13}^I + 2\bar{A}_{55}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{\bar{\alpha}^2},$$

$$\delta_y^I = \frac{(\bar{A}_{23}^I + 2\bar{A}_{44}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{\bar{\alpha}^2},$$

$$\delta_z^I = \frac{(\bar{A}_{12}^I + 2\bar{A}_{66}^I - \bar{\alpha}^2(Q_P^{\text{iso}})^{-1})}{\bar{\alpha}^2}. \quad \text{(A-9)}$$

The value Q_P^{iso} is a quality factor corresponding to an isotropic dissipative medium. The value $\hat{Q}_P$ reduces to Q_P^{iso} if parameters A-9 are zero. By inserting equation A-9 into equation A-7 or A-8, we can easily prove that $\hat{Q}_P$ does not depend on Q_P^{iso} if parameters A-9 are nonzero.

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