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Time-averaged and time-dependent energy-related quantities of harmonic waves in inhomogeneous viscoelastic anisotropic media

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SUMMARY

The energy–flux vector and other energy-related quantities play an important role in various wave propagation problems. In acoustics and seismology, the main attention has been devoted to the time-averaged energy flux of time-harmonic wavefields propagating in non-dissipative, isotropic and anisotropic media. In this paper, we investigate the energy–flux vector and other energy-related quantities of wavefields propagating in inhomogeneous anisotropic viscoelastic media. These quantities satisfy energy-balance equations, which have, as we show, formally different forms for real-valued wavefields with arbitrary time dependence and for time-harmonic wavefields. In case of time-harmonic wavefields, we study both time-averaged and time-dependent constituents of the energy-related quantities. We show that the energy-balance equations for time-harmonic wavefields can be obtained in two different ways. First, using real-valued wavefields satisfying the real-valued equation of motion and stress–strain relation. Second, using complex-valued wavefields satisfying the complex-valued equation of motion and stress–strain relation. The former approach yields simple results only for particularly simple viscoelastic models, such as the Kelvin–Voigt model. The latter approach is considerably more general and can be applied to viscoelastic models of unrestricted anisotropy and viscoelasticity. Both approaches, when applied to the Kelvin–Voigt viscoelastic model, yield the same expressions for the time-averaged and time-dependent constituents of all energy-related quantities and the same energy-balance equations. This indicates that the approach based on complex-valued representation of the wavefield may be used for time harmonic waves quite universally. This study also shows importance of joint consideration of time-averaged and time-dependent constituents of the energy-related quantities in some applications.

Key words: inhomogeneous media, seismic anisotropy, seismic waves, viscoelasticity.

1 INTRODUCTION

The energy–flux vector is one of the basic characteristics of seismic wave fields propagating in the Earth's interior, which plays an important role in many applications in seismology and seismic exploration; both theoretical and practical. In this paper, the properties of the energy–flux vector and of other energy-related quantities of waves propagating in anisotropic dissipative media are studied. Only dissipative mechanisms, which can be described within the framework of linear viscoelasticity, are considered. All derivations are based on the equation of motion and on appropriate stress–strain relation.

Extensive literature has been devoted to the energy flux of wavefields propagating in perfectly elastic, isotropic and anisotropic, media and in viscoelastic isotropic media. For anisotropic viscoelastic media, however, the references are not as numerous. For an up-to-date reviews related to this problem see Carcione (2001) and Červený & Pšenčík (2006). In spite of rapid progress in this field, certain problems still remain open. In this paper, we discuss several of them.

The relations between the energy–flux vector and other energy-related quantities (kinetic, strain and dissipated energy and energy supplied by internal sources) are known in the literature under many names: energy-balance equation, energy-conservation equation, law on conservation of energy, energy-conservation theorem, Poynting theorem, etc. Throughout this paper, we shall consistently use the term *energy-balance equation*.

When studying energy-balance equations, two forms of wavefields are considered: (1) real-valued wavefields of arbitrary time dependence with real-valued stress–strain relations and (2) real-valued or complex-valued time-harmonic wavefields with complex-valued stress–strain relations. The energy-balance equations for these two cases are formally different. For example, the energy-balance equation for wavefields with arbitrary time dependence contains the time-derivative of the *sum* of the kinetic and strain (potential) energy, whereas the energy-balance equation for time-harmonic wavefields contains the *difference* between the kinetic and potential energy (the Lagrangian). An attempt to explain these differences is made in this paper.

For real-valued wavefields of arbitrary time dependence, the stress–strain relation is expressed in convolutory form. To avoid possible complications connected with this general convolutory form, we can consider a simple anisotropic viscoelastic model, called here the Kelvin–Voigt model, in which the stress–strain relation reduces to standard multiplication. It is well known that the Kelvin–Voigt model is not suitable for the investigation of seismic wave propagation in rocks. Notwithstanding, the Kelvin–Voigt stress–strain relation is quite useful for the methodical investigations whose purpose is the derivation of results valid for models of unrestricted anisotropy and viscoelasticity, see Section 4. The energy-balance equation, which gives the relation between the divergence of the energy flux vector and the time derivative of the sum of kinetic and strain energy, the dissipated energy and the energy supplied by internal sources, is derived. For non-dissipative media, the energy-balance equation reduces to the energy-conservation law, well known from seismological textbooks, see, for example, Auld (1973), Carcione (2001) and Červený (2001a). All the energy-related quantities, including the energy flux, are real valued and time dependent in this law.

We focus our attention mainly on time-harmonic, real- or complex-valued wave fields. There are two approaches to deriving the energy-balance equations for time-harmonic waves:

(a) We specify the real-valued, time-dependent, energy-balance equation for real-valued, but time-harmonic wavefields. This is, of course, possible, as the energy-balance equation is valid for wavefields with arbitrary time dependence. Actually, two such real-valued energy-balance equations can be derived. The two energy-balance equations can be combined into one complete, complex-valued energy-balance equation. All energy quantities in this complex-valued energy-balance equation are expressed as a sum of their time-averaged and time-dependent constituents.

(b) We start directly from the complex-valued equation of motion and complex-valued stress–strain relation, valid for time-harmonic waves. We obtain two energy-balance equations, one for time-averaged energy quantities, and the other for time-dependent energy quantities, and combine them again into one complete complex-valued energy-balance equation. This equation is not limited to the Kelvin–Voigt viscoelastic model, but is valid for models of unrestricted viscoelasticity and anisotropy.

The complete complex-valued energy-balance equations, derived by the above two independent approaches for time-harmonic waves, fully coincide if both are specified for the Kelvin–Voigt viscoelastic model. The expressions for all time-averaged and time-dependent energy-related quantities are then the same in both approaches. From this we conclude that both approaches are replaceable for time-harmonic waves and that the second approach, which is simpler and valid for models of unrestricted viscoelasticity and anisotropy, can be used quite generally.

In the seismological literature, the main attention has been devoted to the time-averaged energy–flux vector and energy-related quantities. The studies of the time-dependent energy flux are more or less exceptional. Basic references to time-averaged and time-dependent energy quantities and energy-balance equations of waves propagating in viscoelastic isotropic and anisotropic media are Carcione & Cavallini (1993), Carcione (1999, 2001). Certain equations presented in this paper were derived, in a different form, already in Carcione's publications. Other studies of time-dependent energy flux are related mostly to non-dissipative media (Hayes 1980; Mann *et al.* 1987; Boulanger & Hayes 2000). For non-dissipative media, it has been common to speak of the instantaneous energy flux. As the stress–strain relations for viscoelastic media are not, in general, instantaneous, we prefer to speak of the time-dependent energy flux in dissipative media. A most thorough treatment of the time-dependent energy flux in non-dissipative fluid media, can be found in Mann *et al.* (1987), who also investigated the trains of waves. They showed that consideration of the time-dependent energy flux is a necessary condition for the study of phenomena like an acoustic vortex. Some other authors discussed the time-dependent energy flux in relation to the reflection/transmission problem at a plane interface between two viscoelastic media. See Deschamps (1990), Ainslie & Burns (1995) and Červený (2001b). Ainslie & Burns (1995) call the time-dependent energy–flux vector Deschamp's instantaneous pseudo-energy vector, Červený (2001b) calls it the instantaneous energy–flux vector or pseudo-energy flux vector. The solution of the reflection/transmission problem at a plane interface between two viscoelastic media is complicated by the appearance of interaction energy contributions, generated by the interference of the individual waves, involved in the R/T process. Červený (2001b) showed that the interaction contributions completely vanish if only time-dependent constituents of individual waves (incident, reflected and transmitted; P, S1 and S2) are considered.

In Section 2, the basic equations describing wave propagation in inhomogeneous viscoelastic anisotropic media are introduced and the stress–strain relations are briefly discussed. In Section 3, the real-valued wavefields are considered. A simple real-valued viscoelastic anisotropic stress–strain relation is introduced in Section 3.1, called here the Kelvin–Voigt stress–strain relation. This stress–strain relation is used in Section 3.2 to obtain a real-valued energy-balance equation for wavefields with arbitrary time-dependence. In Section 3.3, two real-valued energy-balance equations for real-valued time-harmonic wave fields in the Kelvin–Voigt model are derived. Their time-averaged and time-dependent constituents are considered. Both real-valued energy-balance equations are combined into one complete complex-valued energy-balance equation. Note that the simple and transparent Kelvin–Voigt viscoelastic anisotropic model considered in Section 3 is used here primarily for methodical reasons, namely to show that the approach based on real-valued time-harmonic wavefields satisfying real-valued stress–strain relation and the approach based on complex-valued time-harmonic wavefields satisfying complex-valued stress–strain relation can be used alternatively in viscoelastic anisotropic media. In Section 4, the main results of this paper are presented. The complex-valued energy-balance equations for time-averaged and time-dependent energy quantities in media of unrestricted viscoelasticity and anisotropy are derived directly from complex-valued equation of motion for time-harmonic waves, and from the relevant complex-valued stress–strain relation. These equations are then applied to the Kelvin–Voigt model and the results are compared with those obtained in Section 3. The results of this comparison are discussed in Section 5. Section 5 also contains discussion of different forms of energy-balance equations for time-harmonic wavefields and for wavefields with arbitrary time dependence.

We would like to emphasize that several important equations presented in this paper have been previously published by other authors. Our goal is not to rederive these equations, but to analyse different ways of derivation of energy-balance equations satisfied by energy-related quantities and to clarify the relations of time-averaged and time-dependent constituents of these quantities for time-harmonic waves.

The dot above a letter denotes the time derivative ($\dot{u}_i = \partial u_i/\partial t$), the index following the comma in the subscript denotes the partial derivative with respect to the relevant Cartesian coordinate ($u_{i,j} = \partial u_i/\partial x_j$), and the asterisk in the superscript denotes complex conjugacy. The lower-case indices $i, j, \ldots, m, n$ take values 1, 2, 3; the lower-case indices α, β values 1, 2, ..., 6. The Einstein summation convention over repeated indices is used throughout the paper.

2 BASIC FORMULAE

We use Cartesian coordinates x_i, time t, and denote the Cartesian components of the displacement vector $u_i(x_k, t)$, of the particle-velocity vector $v_i(x_k, t) = \dot{u}_i(x_k, t)$, of the stress tensor $\tau_{ij}(x_k, t)$, and of the strain tensor $e_{ij}(x_k, t)$. The strain tensor e_{ij} can be expressed in terms of the displacement vector u_i as follows:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1}$$

As we shall mostly work with particle-velocity vector v_i instead of the displacement vector u_i, we shall also use the time derivative of eq. (1):

$$\dot{e}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{2}$$

The equation of motion is given by the relation

$$\tau_{ij,j} + f_i = \rho \dot{v}_i, \tag{3}$$

where $\rho(x_k)$ is the density and $f_i(x_k, t)$ is the density of the body force vector representing internal sources.

Eqs (2) and (3) must be supplemented by a stress–strain relation, connecting τ_{ij} and e_{ij}. Various anelastic material processes influence this relation (grain defects, grain-boundary processes, thermoelastic effects, etc.). A large number of papers is devoted to these anelastic processes. Many of them can be discussed within the framework of linear viscoelasticity, within which stress depends linearly on the strain and its time variation. The stress–strain relation is then expressed in convolutory form, with the relaxation and creep function as the main ingredients.

We start our study by considering one of the simplest stress–strain relations of viscoelastic anisotropic media, in which the convolutory form is reduced to standard multiplication and summation: Kelvin–Voigt stress–strain relation. We use it first for the derivation of the energy-balance equation for real-valued wavefields with arbitrary time dependence. In the rest of the paper, we are mostly interested in the energy-related quantities and energy-balance equations for time-harmonic wavefields, which may be specified in a real- or complex-valued form. The real-valued specification is used throughout the Section 3. For simplicity, we use here the Kelvin–Voigt relation, as it gives clear and transparent results. The complex-valued specification is used in Section 4. In contrast to the real-valued specification, it is applicable to media of unrestricted viscoelasticity and anisotropy.

3 ENERGY-BALANCE EQUATIONS FOR REAL-VALUED WAVE FIELDS AND STRESS–STRAIN RELATIONS

In this section, we introduce a simple real-valued stress–strain relation, for which the time-averaged and time-dependent energy quantities and relevant energy-balance equations in a real-valued form can be derived and discussed without difficulties. We introduce it primarily for having a tool for testing more general complex-valued relations derived in Section 4.

3.1 Kelvin–Voigt stress–strain relation

We consider the following relation between τ_{ij} and e_{ij}, see, for example, Auld (1973), Scott (1997), Shuvalov & Scott (1999) and Carcione (2001):

$$\tau_{ij} = c^R_{ijkl}\, e_{kl} + \eta^R_{ijkl} \dot{e}_{kl}. \tag{4}$$

Here c^R_{ijkl} and η^R_{ijkl} are components of four-rank real-valued elasticity and viscosity tensors, respectively. We will also call the individual components c^R_{ijkl} and η^R_{ijkl} the elasticity and viscosity moduli, respectively. They are real-valued and frequency- and time-independent functions of coordinates x_k. The stress–strain relation (4) will allow us to discuss certain important properties of the time-averaged and time-dependent energy-related quantities, in which we are interested, without unnecessary mathematical complications. Eq. (4) represents an extension of the standard isotropic Kelvin–Voigt viscoelastic model, which can be described in terms of the mechanical spring and dashpot acting in parallel, to anisotropy. At the same time, it represents a simplification of the general Kelvin–Voigt model, in which c^R_{ijkl} and η^R_{ijkl} are time-dependent. For simplicity, we shall call eq. (4) the Kelvin–Voigt viscoelastic anisotropic model, or briefly the Kelvin–Voigt model.

We assume that the elasticity and viscosity moduli satisfy the symmetry relations,

$$c^R_{ijkl} = c^R_{jikl} = c^R_{ijlk} = c^R_{klij}, \quad \eta^R_{ijkl} = \eta^R_{jikl} = \eta^R_{ijlk} = \eta^R_{lkij}. \tag{5}$$

Note that the relation $\eta^R_{ijkl} = \eta^R_{klij}$ has not yet been proved to hold generally, see, for example, Carcione (2001). Relations (5) reduce the number of independent components of elasticity and viscosity tensors from 81 to 21. Due to these symmetries, the elasticity and viscosity tensors can also be expressed as symmetric 6×6 matrices (Voigt notation).

The stress and strain in the relation (4) are real-valued and fully specified in time-domain. Considerably more general complex-valued stress–strain relation for complex-valued time-harmonic wavefields is introduced and used in Section 4. In it, the relevant viscoelastic moduli are complex valued and frequency dependent.

The individual quantities introduced in this section are expressed in the following units: displacement components u_i in metres (m), the particle-velocity components v_i in m s^{-1}, density ρ in kg m^{-3}, the components of body forces f_i in newtons per cubic metre (N m^{-3}; kg m^{-2} s^{-2}), the strain components e_{ij} are dimensionless, the stress components τ_{ij} and elasticity moduli c^R_{ijkl} are in Pascals (Pa; kg m^{-1} s^{-2}), and the viscosity moduli η^R_{ijkl} in Pascals × seconds (Pa s; kg m^{-1} s^{-1}).

3.2 Energy-balance equation for wavefields with arbitrary time dependence

In this section, we consider vectors v_i and f_i, and tensors τ_{ij}, e_{ij} to be real valued and time dependent. Multiplying eq. (3) by $-v_i$ and eq. (2) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i \tau_{ij})_{,j} = -\rho v_i \dot{v}_i - \tau_{ij} \dot{e}_{ij} + v_i f_i. \tag{6}$$

Eq. (6) represents a very general form of the energy-balance equation. The physical meaning of all terms in eq. (6), except $-\tau_{ij}\dot{e}_{ij}$, is the same as in perfectly elastic media and will be explained later. The energy-related term $-\tau_{ij}\dot{e}_{ij}$ depends considerably on the stress–strain relation under consideration. The term should contain two parts: one corresponding to the strain energy and the other to the dissipated energy. For the Kelvin–Voigt model described by the stress–strain relation (4), the definitions of the strain and dissipated energies are well known. The separation of the term $-\tau_{ij}\dot{e}_{ij}$ into the strain- and dissipative-energy terms is thus unique and trivial. Inserting eq. (4) into eq. (6) yields

$$(-v_i\tau_{ij})_{,j} = -\rho v_i \dot{v}_i - c^R_{ijkl}\dot{e}_{ij}e_{kl} - \eta^R_{ijkl}\dot{e}_{ij}\dot{e}_{kl} + v_i f_i. \quad (7)$$

This can be expressed in a condensed form:

$$\mathrm{div}\mathbf{F} = -\frac{\partial}{\partial t}(K+U) - P_d + P_s, \quad (8)$$

where

$$F_j = -v_i\tau_{ij}, \quad K = \tfrac{1}{2}\rho v_i v_i, \quad U = \tfrac{1}{2}c^R_{ijkl}e_{ij}e_{kl}$$
$$P_d = \eta^R_{ijkl}\dot{e}_{ij}\dot{e}_{kl}, \quad P_s = v_i f_i. \quad (9)$$

The vector $\mathbf{F}$ and the scalars K, U, P_d and P_s are real valued and time dependent. The meaning of the individual quantities in eq. (9) is as follows: $\mathbf{F}$ is the energy–flux vector (also called the energy-flow vector, power-flow vector, Poynting vector and Poynting-Umov vector). It represents the amount of energy passing through a unit area perpendicular to $\mathbf{F}$ per unit time. K is the density of kinetic energy, U is the density of strain energy (also called the potential energy), P_d is the density of dissipated energy per second and P_s is the density of energy supplied by internal sources per second.

Eq. (8) is the energy-balance equation of an arbitrary time-dependent real-valued wavefield for Kelvin–Voigt viscoelastic anisotropic media, specified by the strain-stress relation (4). For $P_d = 0$, it immediately yields the well-known energy-balance equation for perfectly elastic media, see Červený (2001a, eq. 2.1.38). Let us emphasize that the energy-balance equation (8) contains a term with the time derivative of the sum of kinetic and strain energy, $K + U$. Such a term does not appear in energy-balance equations for time-harmonic wavefields, presented in the following. This may look surprising. We show, however, that all the forms of the energy-balance equations are fully consistent.

Since the density of strain energy U should be positive, and the density of dissipated energy per second, P_d, non-negative, the 6×6 Voigt elasticity matrix must be positive-definite, and the 6×6 Voigt viscosity matrix positive definite or zero.

The individual energy quantities in eq. (9) have the following dimensions: U and K are expressed in Pascals (Pa; kg m^{-1} s^{-2}), P_d and P_s in Pascals per second (Pa s^{-1}; kg m^{-1} s^{-3}), and $\mathbf{F}$ in watts per square metre (W m^{-2}; kg s^{-3}). Thus, all terms in the energy-balance eq. (8) are expressed in Pascals per second.

3.3 Energy-balance equations for real-valued time-harmonic wavefields

Let us now use the energy-balance equation (8) for time-harmonic waves with frequency $\omega > 0$. In Sections 3.3.1 and 3.3.2, we use two different specifications of time-harmonic real-valued expressions for v_i, f_i, τ_{ij}, e_{ij}, describing the time-harmonic waves. For the derivation of the complete complex-valued energy-balance equation, both these specifications are required, see Section 3.3.3.

3.3.1 Energy-balance equation for Re $\mathbf{F}^+$

We express the real-valued time-dependent quantities v_i, f_i, τ_{ij}, e_{ij} as follows:

$$v_i = \tfrac{1}{2}\left(V_i e^{\mathrm{i}\omega t} + V_i^* e^{-\mathrm{i}\omega t}\right), \quad e_{ij} = \tfrac{1}{2}\left(E_{ij}e^{\mathrm{i}\omega t} + E^*_{ij}e^{-\mathrm{i}\omega t}\right),$$
$$f_i = \tfrac{1}{2}\left(\mathcal{F}_i e^{\mathrm{i}\omega t} + \mathcal{F}_i^* e^{-\mathrm{i}\omega t}\right), \quad \tau_{ij} = \tfrac{1}{2}\left(T_{ij}e^{\mathrm{i}\omega t} + T^*_{ij}e^{-\mathrm{i}\omega t}\right). \quad (10)$$

Here V_i, $\mathcal{F}_i$, E_{ij} and T_{ij} are complex-valued, time-independent, frequency-dependent quantities. We now insert eq. (10) into eq. (7). For example, for $-v_i\tau_{ij}$, we obtain:

$$\begin{aligned}-v_i\tau_{ij} &= -\frac{1}{4}\left(V_i e^{\mathrm{i}\omega t} + V_i^* e^{-\mathrm{i}\omega t}\right)\left(T_{ij}e^{\mathrm{i}\omega t} + T^*_{ij}e^{-\mathrm{i}\omega t}\right)\\ &= -\frac{1}{4}\left(V_i T^*_{ij} + V_i^* T_{ij}\right) - \frac{1}{4}\left[V_i T_{ij}e^{2\mathrm{i}\omega t} + V_i^* T^*_{ij}e^{-2\mathrm{i}\omega t}\right]\\ &= -\frac{1}{2}\mathrm{Re}\left(V_i^* T_{ij}\right) - \frac{1}{2}\mathrm{Re}\left(V_i T_{ij}e^{2\mathrm{i}\omega t}\right). \end{aligned} \quad (11)$$

Since v_i and τ_{ij} are real valued, the expression for $-v_i\tau_{ij}$ is also real-valued. The first term on the right-hand side of eq. (11) is time-independent, and the second term is time harmonic, with frequency 2ω. We can immediately see that the first term represents the time average of $-v_i\tau_{ij}$ over one period. For such time averaging, the second term is zero.

We introduce complex-valued, frequency-dependent, vectors $\mathbf{F}^+$, $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$,

$$\mathbf{F}^+ = \bar{\mathbf{F}} + \tilde{\mathbf{F}}, \quad (12)$$

where $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ are given by relations

$$\bar{F}_j = -\frac{1}{2}V_i^* T_{ij}, \quad \tilde{F}_j = -\frac{1}{2}V_i T_{ij}e^{2\mathrm{i}\omega t}. \quad (13)$$

Here $\bar{\mathbf{F}}$ is the complex-valued, time-averaged, energy–flux vector, $\tilde{\mathbf{F}}$ is the complex-valued, time-dependent, energy–flux vector, and $\mathbf{F}^+$ is the complete, complex-valued, energy–flux vector. Then eq. (11) can be expressed as follows:

$$-v_i\tau_{ij} = \mathrm{Re}F_j^+. \quad (14)$$

Quite analogous expressions can also be derived for other energy quantities in eq. (7):

$$-\rho v_i\dot{v}_i = 2\omega\mathrm{Im}K^+, \quad -c^R_{ijkl}\dot{e}_{ij}e_{kl} = -2\omega\mathrm{Im}U^+,$$
$$v_i f_i = \mathrm{Re}P_s^+, \quad -\eta^R_{ijkl}\dot{e}_{ij}\dot{e}_{kl} = -\mathrm{Re}P_d^+, \quad (15)$$

where

$$K^+ = \bar{K} + \tilde{K}, \quad U^+ = \bar{U} + \tilde{U},$$
$$P_s^+ = \bar{P}_s + \tilde{P}_s, \quad P_d^+ = \bar{P}_d + \tilde{P}_d. \quad (16)$$

The time-averaged quantities $\bar{K}$, $\bar{U}$, $\bar{P}_s$ and $\bar{P}_d$ are given by relations:

$$\bar{K} = \tfrac{1}{4}\rho V_i V_i^*, \quad \bar{U} = \tfrac{1}{4}c^R_{ijkl}E_{ij}E^*_{kl},$$
$$\bar{P}_s = \tfrac{1}{2}V_i^*\mathcal{F}_i, \quad \bar{P}_d = \tfrac{1}{2}\omega^2\eta^R_{ijkl}E_{ij}E^*_{kl}. \quad (17)$$

The time-dependent quantities $\tilde{K}$, $\tilde{U}$, $\tilde{P}_s$ and $\tilde{P}_d$ are given by relations:

$$\tilde{K} = \tfrac{1}{4}\rho V_i V_i e^{2\mathrm{i}\omega t}, \quad \tilde{U} = -\tfrac{1}{4}c^R_{ijkl}E_{ij}E_{kl}e^{2\mathrm{i}\omega t},$$
$$\tilde{P}_s = \tfrac{1}{2}V_i\mathcal{F}_i e^{2\mathrm{i}\omega t}, \quad \tilde{P}_d = -\tfrac{1}{2}\omega^2\eta^R_{ijkl}E_{ij}E_{kl}e^{2\mathrm{i}\omega t}. \quad (18)$$

We can immediately see that $\bar{K}$, $\bar{U}$ and $\bar{P}_d$ are real valued, the latter two quantities thanks to the symmetries of c^R_{ijkl} and η^R_{ijkl}, see eq. (5). Consequently,

$$\mathrm{Re}\bar{K} = \bar{K}, \quad \mathrm{Re}\bar{U} = \bar{U}, \quad \mathrm{Re}\bar{P}_d = \bar{P}_d,$$
$$\mathrm{Im}\bar{K} = 0, \quad \mathrm{Im}\bar{U} = 0, \quad \mathrm{Im}\bar{P}_d = 0. \quad (19)$$

Inserting eqs (14) and (15) into eq. (7), and taking into account eq. (19), we obtain the real-valued energy-balance equation for time-harmonic waves in the form:

$$\mathrm{div}(\mathrm{Re}\mathbf{F}^+) = 2\omega\mathrm{Im}(\tilde{K} - \tilde{U}) - \mathrm{Re}P_d^+ + \mathrm{Re}P_s^+. \quad (20)$$

It is interesting to note that the real-valued energy-balance equation for Re $\mathbf{F}^+$ does not depend on the time-averaged kinetic and strain energies.

The energy-balance equation (20) is often expressed in its time-averaged form. It reads, see eqs (19) and (20),

$$\mathrm{div}(\mathrm{Re}\bar{\mathbf{F}}) = -\bar{P}_d + \mathrm{Re}\bar{P}_s. \tag{21}$$

The time-averaged energy-balance equation (21) is not new; it was derived and discussed by Auld (1973), Carcione & Cavallini (1993), Carcione (1999, 2001), and Červený & Pšenčík (2006) for general viscoelastic anisotropic media.

The real-valued vector $\mathrm{Re}\bar{\mathbf{F}}$ is sometimes called the active part of the energy–flux vector $\bar{\mathbf{F}}$, see Mann *et al.* (1987). The interpretation of the energy-balance equation (21) is simple. The outwardly directed active part of the energy flux is balanced by the energy supplied by internal sources decreased by the dissipated energy. This energy-balance equation is not influenced at all by kinetic and strain energies.

3.3.2 *Energy-balance equation for* Im $\mathbf{F}^+$

We now express the real-valued, time-dependent quantities v_i, f_i, e_{ij} and τ_{ij} in a way alternative to eq. (10):

$$\hat{v}_i = \tfrac{1}{2\mathrm{i}}(V_i e^{\mathrm{i}\omega t} - V_i^* e^{-\mathrm{i}\omega t}), \quad \hat{e}_{ij} = \tfrac{1}{2\mathrm{i}}(E_{ij} e^{\mathrm{i}\omega t} - E_{ij}^* e^{-\mathrm{i}\omega t}),$$
$$\hat{f}_i = \tfrac{1}{2\mathrm{i}}(\mathcal{F}_i e^{\mathrm{i}\omega t} - \mathcal{F}_i^* e^{-\mathrm{i}\omega t}), \quad \hat{\tau}_{ij} = \tfrac{1}{2\mathrm{i}}(T_{ij} e^{\mathrm{i}\omega t} - T_{ij}^* e^{-\mathrm{i}\omega t}), \tag{22}$$

using the circumflex above letters to distinguish expressions in eqs (22) and (10).

Using eq. (22) in eq. (3), and eq. (10) in eq. (2), we obtain

$$\hat{\tau}_{ij,j} + \hat{f}_i = \rho\dot{\hat{v}}_i, \quad \dot{e}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}). \tag{23}$$

Both eqs (23) are expressed in consistent quantities, so that they can be treated together. Multiplying the first eq. (23) by $-v_i$ and the second by $-\hat{\tau}_{ij}$, adding both, and using eq. (4), yields:

$$(-v_i\hat{\tau}_{ij})_{,j} = -\rho v_i\dot{\hat{v}}_i - c^R_{ijkl}\dot{e}_{ij}\hat{e}_{kl} - \eta^R_{ijkl}\dot{e}_{ij}\dot{\hat{e}}_{kl} + v_i\hat{f}_i. \tag{24}$$

Eq. (24) represents an alternative form of the energy-balance equation (7) specified for time-harmonic waves. The frequency under consideration is again $\omega > 0$. All the terms in eq. (24) are real valued and time dependent. They are given by expressions:

$$-v_i\hat{\tau}_{ij} = \mathrm{Im}F_j^+, \qquad -\rho v_i\dot{\hat{v}}_i = -2\omega\mathrm{Re}K^+,$$
$$-c^R_{ijkl}\dot{e}_{ij}\hat{e}_{kl} = 2\omega\mathrm{Re}U^+, \quad -\eta^R_{ijkl}\dot{e}_{ij}\dot{\hat{e}}_{kl} = -\mathrm{Im}P_d^+,$$
$$v_i\hat{f}_i = \mathrm{Im}P_s^+. \tag{25}$$

Here the complete energy flux F_j^+ is given by eq. (12) with eqs (13), and the complete energy quantities K^+, U^+, P_d^+ and P_s^+ by eqs (16), with eqs (17) and (18). Inserting eqs (25) into eq. (24), and taking into account that $\mathrm{Im}\bar{P}_d = 0$, see eqs (19), we obtain the real-valued energy-balance equation for Im F_j^+:

$$\mathrm{div}(\mathrm{Im}\mathbf{F}^+) = -2\omega\mathrm{Re}(K^+ - U^+) - \mathrm{Im}\tilde{P}_d + \mathrm{Im}P_s^+. \tag{26}$$

The energy-balance equation (26) does not depend on the time-averaged dissipated energy. Analogously to eq. (21), eq. (26) yields the energy-balance equation for $\mathrm{Im}\bar{\mathbf{F}}$ in the time-averaged form,

$$\mathrm{div}(\mathrm{Im}\bar{\mathbf{F}}) = -2\omega(\bar{K} - \bar{U}) + \mathrm{Im}\bar{P}_s. \tag{27}$$

Similarly as eq. (21), also the energy-balance equation (27) for time-averaged energy-related quantities is well known; see the references below (21). Let us note that $\mathrm{Im}\bar{\mathbf{F}}$ represents a real-valued, time-averaged, energy-related vector, which has various applications. For example, eq. (27) represents an important relation between $\mathrm{Im}\bar{\mathbf{F}}$ and the real-valued time-averaged energy quantities $\bar{K}$ and $\bar{U}$ in a viscoelastic anisotropic medium. Similar conclusions apply to the real-valued time-dependent vector $\mathrm{Im}\tilde{\mathbf{F}}$ and to the real-valued vector $\mathrm{Im}\mathbf{F}^+ = \mathrm{Im}\bar{\mathbf{F}} + \mathrm{Im}\tilde{\mathbf{F}}$. This is one of the reasons why the Poynting vector $\mathbf{F}$ is often considered complex-valued, mainly in acoustics.

The imaginary part of $\bar{\mathbf{F}}$ is sometimes called the reactive part of the energy flux vector $\bar{\mathbf{F}}$, see Mann *et al.* (1987). Eq. (27) allows the following interpretation: The reactive part of the energy–flux vector is generated by the difference of the time-averaged strain and kinetic energies and by the imaginary part of the time-averaged energy supplied by internal sources per second. In perfectly elastic media, where $\bar{K} = \bar{U}$, in the absence of internal sources, divergence of the reactive part of $\bar{\mathbf{F}}$ vanishes.

3.3.3 *Complete complex-valued energy-balance equation*

Let us combine both real-valued energy-balance equations for Re $\mathbf{F}^+$ and Im $\mathbf{F}^+$, eqs (20) and (26). Multiplying eq. (26) by the imaginary unit and adding it to eq. (20), we obtain

$$\mathrm{div}\mathbf{F}^+ = -2\mathrm{i}\omega(K^+ - U^+) - P_d^+ + P_s^+. \tag{28}$$

We call eq. (28) the complete, complex-valued, energy-balance equation for time-harmonic waves. All energy-related quantities in eq. (28) are complex valued, and are given by eqs (12) and (16). They include both time-averaged and time-dependent energy constituents.

The time average of the complete complex-valued energy-balance eq. (28) reads

$$\mathrm{div}\bar{\mathbf{F}} = -2\mathrm{i}\omega(\bar{K} - \bar{U}) - \bar{P}_d + \bar{P}_s. \tag{29}$$

Remember that $\mathrm{Im}\bar{K} = \mathrm{Im}\bar{U} = \mathrm{Im}\bar{P}_d = 0$ in eq. (29), see eqs (19). The notes below eq. (21), including the references given there, hold also for eq. (29).

3.3.4 *Time-dependent energy-related quantities*

The time-dependent energy-related quantities $\tilde{K}$, $\tilde{U}$, $\tilde{P}_s$ and $\tilde{P}_d$ are given by eqs (18). They can be expressed in many alternative forms. For example, we can introduce time-independent arguments (phase shifts) χ_K, χ_U, χ_s and χ_d of complex-valued quantities V_iV_i, $V_i\mathcal{F}_i$, $c^R_{ijkl}E_{ij}E_{kl}$ and $\eta^R_{ijkl}E_{ij}E_{kl}$ as follows:

$$V_iV_i = |V_iV_i|e^{\mathrm{i}\chi_K}, \quad c^R_{ijkl}E_{ij}E_{kl} = c^R_{ijkl}|E_{ij}E_{kl}|e^{\mathrm{i}\chi_U},$$
$$V_i\mathcal{F}_i = |V_i\mathcal{F}_i|e^{\mathrm{i}\chi_s}, \quad \eta^R_{ijkl}E_{ij}E_{kl} = \eta^R_{ijkl}|E_{ij}E_{kl}|e^{\mathrm{i}\chi_d}. \tag{30}$$

The time-dependent energy-related quantities $\tilde{K}$, $\tilde{U}$, $\tilde{P}_s$ and $\tilde{P}_d$ given by eqs (18) then read

$$\tilde{K} = \tfrac{1}{4}\rho|V_iV_i|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_K}, \quad \tilde{U} = -\tfrac{1}{4}c^R_{ijkl}|E_{ij}E_{kl}|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_U},$$
$$\tilde{P}_s = \tfrac{1}{2}|V_i\mathcal{F}_i|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_s}, \quad \tilde{P}_d = -\tfrac{1}{2}\omega^2\eta^R_{ijkl}|E_{ij}E_{kl}|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_d}. \tag{31}$$

As only the exponential functions in eqs (31) are complex valued, we obtain the real and imaginary parts of the energy-related quantities from eqs (31) only by replacing the exponential functions by cosines and sines, respectively. For example,

$$\mathrm{Re}\tilde{K} = \tfrac{1}{4}\rho|V_i V_i|\cos(2\omega t + \chi_K),$$
$$\mathrm{Im}\tilde{K} = \tfrac{1}{4}\rho|V_i V_i|\sin(2\omega t + \chi_K). \tag{32}$$

Let us now discuss the time-dependent constituents of the *complex-valued energy–flux vector* $\tilde{\mathbf{F}}$. The expression for its *j*th Cartesian component reads:

$$\tilde{F}_j = -\frac{1}{2}|V_i T_{ij}|e^{\mathrm{i}(2\omega t + \chi_j)}, \tag{33}$$

where χ_j is given by the relation

$$V_i T_{ij} = |V_i T_{ij}|e^{\mathrm{i}\chi_j}. \tag{34}$$

Vector $\tilde{\mathbf{F}} = \mathrm{Re}\tilde{\mathbf{F}} + \mathrm{iIm}\tilde{\mathbf{F}}$ can be expressed as follows:

$$\mathrm{Re}\tilde{\mathbf{F}} = \mathbf{F}_1\cos 2\omega t - \mathbf{F}_2\sin 2\omega t \quad \mathrm{Im}\tilde{\mathbf{F}} = \mathbf{F}_1\sin 2\omega t + \mathbf{F}_2\cos 2\omega t, \tag{35}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are real-valued and time-independent vectors with components

$$F_{1j} = -\frac{1}{2}\mathrm{Re}(V_i T_{ij}), \quad F_{2j} = -\frac{1}{2}\mathrm{Im}(V_i T_{ij}). \tag{36}$$

3.3.5 *Complete complex-valued energy-related quantities*

Using eqs (16), (17) and (31), we easily obtain the expressions for the complete, complex-valued, energy-related quantities K^+, U^+, P_d^+ and P_s^+.

For the complete, complex-valued, energy–flux vector $\mathbf{F}^+ = \mathrm{Re}\mathbf{F}^+ + \mathrm{iIm}\,\mathbf{F}^+$, we obtain

$$\mathrm{Re}\mathbf{F}^+ = \mathrm{Re}\bar{\mathbf{F}} + \mathbf{F}_1\cos 2\omega t - \mathbf{F}_2\sin 2\omega t,$$
$$\mathrm{Im}\mathbf{F}^+ = \mathrm{Im}\bar{\mathbf{F}} + \mathbf{F}_1\sin 2\omega t + \mathbf{F}_2\cos 2\omega t. \tag{37}$$

Thus, Re $\mathbf{F}^+$ of time-harmonic waves varies periodically with time and frequency 2ω around its time-averaged part $\mathrm{Re}\bar{\mathbf{F}}$. Its behaviour is completely specified by three time-independent vectors, namely $\mathrm{Re}\bar{\mathbf{F}}$, $\mathbf{F}_1$ and $\mathbf{F}_2$. Similarly, the oscillations of $\mathrm{Im}\mathbf{F}^+$ are fully specified by three time-independent vectors $\mathrm{Im}\bar{\mathbf{F}}$, $\mathbf{F}_1$ and $\mathbf{F}_2$. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel, the oscillations are linear. If $\mathbf{F}_1$ and $\mathbf{F}_2$ make a non-zero angle, the oscillations are elliptical.

In various applications, it is useful to know the complete complex-valued energy flux through a plane specified by normal $\mathbf{n}$. We obtain

$$\mathrm{Re}\mathbf{F}^+\cdot\mathbf{n} = -\tfrac{1}{2}\mathrm{Re}(V_i^* T_{ij} n_j) - \tfrac{1}{2}|V_i T_{ij} n_j|\cos(2\omega t + \chi),$$
$$\mathrm{Im}\mathbf{F}^+\cdot\mathbf{n} = -\tfrac{1}{2}\mathrm{Im}(V_i^* T_{ij} n_j) - \tfrac{1}{2}|V_i T_{ij} n_j|\sin(2\omega t + \chi), \tag{38}$$

where

$$\cos\chi = \frac{\mathrm{Re}(V_i T_{ij} n_j)}{|V_i T_{ij} n_j|}, \quad \sin\chi = \frac{\mathrm{Im}(V_i T_{ij} n_j)}{|V_i T_{ij} n_j|}. \tag{39}$$

Eqs (38) and (39) may play an important role in the solution of the reflection/transmission problem on a plane interface. They can be used to decide whether the energy flux of the wave under consideration is pointing away from or towards the interface. Actually, according to eqs (38) it may be pointing away from the interface at some times, and to the interface at other times within a period.

3.3.6 *Peak values of energy-related quantities*

Under peak values we understand the maximum values of energy-related quantities as a function of time *t*. These values, of course, differ from the time-averaged values, but as them, they are time independent. They are composed of the time-averaged value of the corresponding quantity and the peak value of its time-dependent constituent. In the following, we only present the expressions for the latter values.

From eqs (31), we obtain

$$\tilde{K}^{\mathrm{peak}} = \tfrac{1}{4}\rho|V_i V_i|, \quad \tilde{U}^{\mathrm{peak}} = \tfrac{1}{4}\omega c^R_{ijkl}|E_{ij}E_{kl}|,$$
$$\tilde{P_s}^{\mathrm{peak}} = \tfrac{1}{2}|V_i\mathcal{F}_i|, \quad \tilde{P_d}^{\mathrm{peak}} = \tfrac{1}{2}\omega^2\eta^R_{ijkl}|E_{ij}E_{kl}|. \tag{40}$$

We can see that all the above peak values are real valued. The peak values are the same for the complex-valued energy-related quantities and their real and imaginary parts.

For the *j*th component $\tilde{F}_j$ of the time-dependent constituent of the complex-valued energy–flux vector, we obtain from eq. (33),

$$\tilde{F}_j^{\mathrm{peak}} = \frac{1}{2}|V_i T_{ij}|. \tag{41}$$

We can also specify the peak value of the time-dependent constituent of the energy flux through a plane specified by normal $\mathbf{n}$,

$$(\tilde{\mathbf{F}}\cdot\mathbf{n})^{\mathrm{peak}} = \frac{1}{2}|V_i T_{ij} n_j|. \tag{42}$$

4 DIRECT DERIVATION OF ENERGY-BALANCE EQUATION FOR COMPLEX-VALUED TIME-HARMONIC WAVEFIELDS

In the previous sections, we derived the energy-balance equations for time-harmonic waves considering two real-valued representations of the wavefield. In their derivation, we considered the particularly simple case of the stress–strain relation, corresponding to the Kelvin–Voigt model, see eq. (4).

In this section, we consider the complex-valued representation of the time-harmonic wavefield, and the complex-valued frequency-dependent stress–strain relation for an arbitrary viscoelastic model. In spite of this generalization, the derivation is simpler and more transparent, and directly combines both approaches, used independently in Section 3. We compare the results we obtain here for general viscoelasticity and the complex-valued representation of the wavefield, with those for the Kelvin–Voigt viscoelastic model and the real-valued representation of the wavefield, obtained in Section 3. We shall see that both approaches yield exactly the same final equations for the Kelvin–Voigt viscoelastic model. This indicates that the general approach, presented in this section, based on the complex-valued representation of the wavefield, may be used as a generalization of the approach based on the real-valued representation of the wavefield.

In the following, we consider vectors v_i, f_i and tensors τ_{ij}, e_{ij} to be complex-valued time-harmonic quantities:

$$v_i = V_i e^{\mathrm{i}\omega t}, \quad \tau_{ij} = T_{ij}e^{\mathrm{i}\omega t},$$
$$e_{ij} = E_{ij}e^{\mathrm{i}\omega t}, \quad f_i = \mathcal{F}_i e^{\mathrm{i}\omega t}. \tag{43}$$

We use the correspondence principle (Carcione 2001). According to it the equation of motion, the time derivative of the strain tensor and the stress–strain relation in viscoelastic anisotropic media, that is,

$$\tau_{ij,j} + f_i = \rho\dot{v}_i, \quad \dot{e}_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \tag{44}$$

and

$$\tau_{ij} = c_{ijkl}e_{kl} \tag{45}$$

have the same form as in perfectly elastic anisotropic media; only the moduli c_{ijkl} in eq. (45) and the quantities V_i, $\mathcal{F}_i$, T_{ij} and E_{ij} in eqs (43) are complex valued and frequency dependent.

4.1 Complex-valued frequency-dependent stress–strain relation

For time-harmonic waves propagating in a linear, viscoelastic, anisotropic medium, the complex-valued, frequency-dependent stress–strain relation is given by eq. (45), where

$$c_{ijkl} = \mathrm{Re}[c_{ijkl}(\omega)] + \mathrm{i}\,\mathrm{Im}[c_{ijkl}(\omega)]. \tag{46}$$

We will tacitly assume that c_{ijkl} are frequency-dependent, and omit argument ω in the following. We assume that c_{ijkl} satisfy the symmetry relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}, \tag{47}$$

which reduce the number of independent complex-valued moduli from 81 to 21. Thus, we can construct the 6×6 symmetric Voigt viscoelasticity matrices $\mathrm{Re}(C_{\alpha\beta})$ and $\mathrm{Im}(C_{\alpha\beta})$. The proof that $\mathrm{Im}(C_{\alpha\beta}) = \mathrm{Im}(C_{\beta\alpha})$ has not yet been given in the literature, but this problem is not discussed here. We further assume that the 6×6 Voigt matrices $\mathrm{Re}(C_{\alpha\beta})$ and $\mathrm{Im}(C_{\alpha\beta})$ are positive definite and non-negative definite, respectively, for $\omega > 0$ under consideration.

As a special case, necessary for comparison of general results of this section with similar results of Section 3, consider the stress–strain relation (4) for the Kelvin–Voigt viscoelastic model. Taking into account eqs (43), eq. (4) yields

$$\tau_{ij} = \left(c^R_{ijkl} + \mathrm{i}\omega\eta^R_{ijkl}\right)e_{kl}. \tag{48}$$

Consequently, the complex-valued, frequency-dependent viscoelastic anisotropic moduli c_{ijkl} for the Kelvin–Voigt model, see eq. (4), are given by the relation

$$c_{ijkl} = c^R_{ijkl} + \mathrm{i}\omega\eta^R_{ijkl}. \tag{49}$$

The real part of c_{ijkl} is frequency independent, and the imaginary part of c_{ijkl} depends linearly on frequency. Let us emphasize that the Kelvin–Voigt model, specified by moduli (49), represents a considerable simplification of the general model of the viscoelastic anisotropic medium, specified by complex-valued, frequency dependent viscoelastic moduli (46), treated in the following.

4.2 Energy-balance equation for time-averaged quantities

Multiplying the first equation of eqs (44) by $-v_i^*$ and the complex-conjugate of the second equation of eqs (44) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i^*\tau_{ij})_{,j} = -\rho v_i^*\dot{v}_i - \tau_{ij}\dot{e}^*_{ij} + v_i^* f_i. \tag{50}$$

We insert the stress–strain relation (45) into eq. (50), perform the time derivatives, and multiply it by $\frac{1}{2}$, to get :

$$\left(-\frac{1}{2}v_i^*\tau_{ij}\right)_{,j} = -\frac{1}{2}\mathrm{i}\omega\rho v_i v_i^* + \frac{1}{2}\mathrm{i}\omega c_{ijkl}e^*_{ij}e_{kl} + \frac{1}{2}v_i^* f_i. \tag{51}$$

Expressing c_{ijkl} using eq. (46), we obtain,

$$\mathrm{div}\bar{\mathbf{F}} = -2\mathrm{i}\omega(\bar{K} - \bar{U}) - \bar{P}_d + \bar{P}_s, \tag{52}$$

where

$$\begin{aligned}
\bar{F}_j &= -\tfrac{1}{2}v_i^*\tau_{ij} = -\tfrac{1}{2}V_i^*T_{ij}, \quad \bar{K} = \tfrac{1}{4}\rho v_i v_i^* = \tfrac{1}{4}\rho V_i^* V_i,\\
\bar{U} &= \tfrac{1}{4}\mathrm{Re}(c_{ijkl})e^*_{ij}e_{kl} = \tfrac{1}{4}\mathrm{Re}(c_{ijkl})E^*_{ij}E_{kl},\\
\bar{P}_d &= \tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl})e^*_{ij}e_{kl} = \tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl})E^*_{ij}E_{kl},\\
\bar{P}_s &= \tfrac{1}{2}v_i^* f_i = \tfrac{1}{2}V_i^*\mathcal{F}_i.
\end{aligned} \tag{53}$$

Here quantities $\bar{\boldsymbol{F}}$ and $\bar{P}_s$ are complex valued, $\bar{K}$, $\bar{U}$ and $\bar{P}_d$ real valued. Eq. (52) is exactly the same as energy-balance equation (29) for the time-averaged energy quantities, and is well known from seismological literature, see, for example, Carcione & Cavallini (1993), Carcione (1999, 2001) and Červený & Pšenčík (2006).

4.3 Energy-balance equation for time-dependent quantities

Multiplying the first equation of eqs (44) by $-\,v_i$ and the second equation of eqs (44) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i\tau_{ij})_{,j} = -\rho v_i\dot{v}_i - \tau_{ij}\dot{e}_{ij} + v_i f_i. \tag{54}$$

This equation is formally the same as eq. (6), but the individual quantities are not real valued, but complex valued and frequency dependent. Taking the time derivatives in eq. (54), inserting the stress–strain relation (45) and multiplying the result by $\frac{1}{2}$, we get:

$$\mathrm{div}\tilde{\mathbf{F}} = -2\mathrm{i}\omega(\tilde{K} - \tilde{U}) - \tilde{P}_d + \tilde{P}_s. \tag{55}$$

Here

$$\begin{aligned}
\tilde{F}_j &= -\frac{1}{2}v_i\tau_{ij} = -\frac{1}{2}V_iT_{ij}e^{2\mathrm{i}\omega t},\\
\tilde{K} &= \frac{1}{4}\rho v_i v_i = \frac{1}{4}\rho V_i V_i e^{2\mathrm{i}\omega t},\\
\tilde{U} &= -\frac{1}{4}\mathrm{Re}(c_{ijkl})e_{ij}e_{kl} = -\frac{1}{4}\mathrm{Re}(c_{ijkl})E_{ij}E_{kl}e^{2\mathrm{i}\omega t},\\
\tilde{P}_d &= -\frac{1}{2}\omega\mathrm{Im}(c_{ijkl})e_{ij}e_{kl} = -\frac{1}{2}\omega\mathrm{Im}(c_{ijkl})E_{ij}E_{kl}e^{2\mathrm{i}\omega t},\\
\tilde{P}_s &= \frac{1}{2}v_i f_i = \frac{1}{2}V_i\mathcal{F}_i e^{2\mathrm{i}\omega t}.
\end{aligned} \tag{56}$$

4.4 Complete energy-balance equation

In Sections 4.2 and 4.3, we derived two energy-balance equations: eq. (52) for time-averaged energy-related quantities and eq. (55) for time-dependent energy-related quantities. It is natural to combine these two equations to get the complete energy-balance equation, containing both the time-averaged and time-dependent energy-related constituents. Summing up eqs (52) and (55), we obtain the complete energy-balance equation

$$\mathrm{div}\mathbf{F}^+ = -2\mathrm{i}\omega(K^+ - U^+) - P^+_d + P^+_s. \tag{57}$$

Here the energy-related quantities with '+' in the superscript again denote the sum of the time-averaged and time-dependent energy-related quantities. Using eqs (53) and (56), we obtain:

$$\begin{aligned}
F^+_j &= -\frac{1}{2}v_i^*\tau_{ij} - \frac{1}{2}v_i\tau_{ij} = -\mathrm{Re}(v_i)\tau_{ij},\\
K^+ &= \frac{1}{4}\rho v_i v_i^* + \frac{1}{4}\rho v_i v_i = \frac{1}{2}\rho\mathrm{Re}(v_i)v_i,\\
U^+ &= \frac{1}{4}\mathrm{Re}(c_{ijkl})e^*_{ij}e_{kl} - \frac{1}{4}\mathrm{Re}(c_{ijkl})e_{ij}e_{kl}\\
&= -\frac{1}{2}\mathrm{i}\mathrm{Re}(c_{ijkl})\mathrm{Im}(e_{ij})e_{kl},\\
P^+_d &= \frac{1}{2}\omega\mathrm{Im}(c_{ijkl})e^*_{ij}e_{kl} - \frac{1}{2}\omega\mathrm{Im}(c_{ijkl})e_{ij}e_{kl}\\
&= -\mathrm{i}\omega\mathrm{Im}(c_{ijkl})\mathrm{Im}(e_{ij})e_{kl},\\
P^+_s &= \frac{1}{2}v_i^* f_i + \frac{1}{2}v_i f_i = \mathrm{Re}(v_i)f_i.
\end{aligned} \tag{58}$$

We call $\mathbf{F}^+$ the complete energy–flux vector, and K^+, U^+, P^+_d and P^+_s the complete energy-related quantities. It is interesting to note the simplicity of the expressions for the complete energy-related quantities, particularly for $\mathbf{F}^+$, K^+ and P_s^+.

Let us briefly discuss the complete energy flux $\mathbf{F}^+$ from the physical point of view. It is given by the relation, see eqs (58),

$$F_j^+ = -\mathrm{Re}(v_i)\tau_{ij}. \qquad (59)$$

This immediately yields

$$\mathrm{Re}F_j^+ = -\mathrm{Re}(v_i)\mathrm{Re}(\tau_{ij}), \quad \mathrm{Im}F_j^+ = -\mathrm{Re}(v_i)\mathrm{Im}(\tau_{ij}). \qquad (60)$$

Both expressions in eqs (60) are real valued, and contain the time-averaged and time-dependent constituents, see eqs (58). The interpretation of eqs (60) is simple: In order to obtain physically meaningful quantities in energy considerations, the complex-valued quantities must be replaced by their real-valued parts according to eqs (60). In seeking Re F_j^+, one should replace the complex-valued term $-v_i\tau_{ij}$ by $-\mathrm{Re}(v_i)\mathrm{Re}(\tau_{ij})$, in seeking Im F_j^+, one should replace $-v_i\tau_{ij}$ by $-\mathrm{Re}(v_i)\mathrm{Im}(\tau_{ij})$.

In the absence of internal sources ($f_i = 0$), the first equation of eqs (44) yields $\tau_{ik,k} = \rho\dot{v}_i$, so that $v_i = -\mathrm{i}\tau_{ik,k}/\rho\omega$. Then the complete energy–flux vector can be expressed as a function of the stress field only,

$$F_j^+ = \mathrm{Im}(\tau_{ik,k})\tau_{ij}/\rho\omega. \qquad (61)$$

For plane waves propagating in homogeneous viscoelastic anisotropic media, $\tau_{ik,k} = \mathrm{i}\omega\tau_{ik}p_k$. Eq. (61) then reduces to

$$F_j^+ = \mathrm{Re}(\tau_{ik}p_k)\tau_{ij}/\rho. \qquad (62)$$

In eq. (62), $p_{\hat{k}}$ denotes kth component of the slowness vector.

4.5 Alternative complex-valued representation

Alternatively to the complex-valued expressions in eqs (43), expressions containing the exponential time factor $\exp[-\mathrm{i}\omega t]$ have also been often used:

$$v_i = V_i e^{-\mathrm{i}\omega t}, \qquad \tau_{ij} = T_{ij}e^{-\mathrm{i}\omega t},$$
$$e_{ij} = E_{ij}e^{-\mathrm{i}\omega t}, \qquad f_i = \mathcal{F}_i e^{-\mathrm{i}\omega t}. \qquad (63)$$

These expressions are used particularly in the seismic ray theory, see, for example, Červený (2001a).

Without detailed treatment, we show the changes introduced by new complex-valued expressions in eqs (63) with respect to equations described in Sections 4.1–4.4.

First, the stress–strain relation (45) remains valid, but Im (c_{ijkl}) has the opposite sign. Consequently, we have to assume that the 6×6 Voigt matrix $-\mathrm{Im}(C_{\alpha\beta})$ is positive definite or zero.

Secondly, the complete energy-balance equation (57) transforms into:

$$\mathrm{div}\boldsymbol{F}^+ = 2\mathrm{i}\omega(K^+ - U^+) - P_d^+ + P_s^+, \qquad (64)$$

where $\boldsymbol{F}^+$, K^+, U^+ and P_s^+ are given by eqs (58). The only formal difference is in P_d^+, whose sign is opposite to that in eqs (58). The negative sign with P_d^+ in eq. (64), however, remains valid, as $\mathrm{Im}(c_{ijkl})$ is of the opposite sign. Note that eq. (64) has been used by Červený & Pšenčík (2006).

Thirdly, the complex-valued energy-related quantities in eq. (64) must be taken complex-conjugate to those in eqs (58).

5 COMPARISON OF COMPLEX- AND REAL-VALUED APPROACHES FOR TIME-HARMONIC WAVES

The approach used in Section 4 to derive energy-balance equations and expressions for individual energy-related quantities for time-harmonic waves is valid for unrestricted anisotropy and viscoelasticity. In the derivation, we considered the complex-valued representation of time-harmonic wavefields, and the complex-valued frequency-dependent stress–strain relations, see eq. (45) with eq. (46). The complete energy-balance equation (57) is, of course, different from the energy-balance equation (6), corresponding to wavefields with arbitrary time dependence. The most problematic term of the energy-balance equation (6) in the time domain is the term $-\tau_{ij}\dot{e}_{ij}$ containing the strain and dissipated energies. To simplify our task, we consider the simple Kelvin–Voigt stress–strain relation (4), for which the definitions of strain and dissipated energies are well known. For it, the energy-balance equation (6) can be expressed in the form of (8). The energy-balance equation (8) is physically very appealing, as it is expressed in the form well known from studies of wave propagation in isotropic dissipative media. It contains the well-known term with the time-derivative of the sum of kinetic and strain energy, $K + U$.

Let us note that the general energy-balance equation (57), derived for time-harmonic waves, does not contain the above-mentioned term with $K + U$. This could lead to a conclusion that there is a conflict between the two energy-balance equations (57) and (8). To show that this is not the case was one of the reasons why we had applied the time-domain energy-balance equation for Kelvin–Voigt viscoelastic anisotropic model to real-valued time-harmonic wavefields in Section 3.3.

Computations performed in Sections 3.3 and 4 show, that the general approach of Section 4, when specified to the Kelvin–Voigt model, gives the same results as the time-domain approach of Section 3.3 (for time-harmonic waves). This is simple to see when comparing the complete energy balance equations (57) with (28) and energy-related quantities (58) with quantities in equations (12), (13) and (16)–(18). Similarly, we can compare the energy-balance equations (52) and (29) for time-averaged quantities, expressions for individual energy-related quantities both in their time-averaged and time-dependent forms, etc. This indicates that for time-harmonic waves the two approaches, when applicable, are fully replaceable and that the general approach of Section 4 can be used quite universally.

6 CONCLUSION

Two approaches are used to investigate the energy–flux vector and other energy-related quantities of time-harmonic wavefields propagating in inhomogeneous anisotropic viscoelastic media. The first approach is limited to very simple viscoelastic media like the Kelvin–Voigt viscoelastic model, and is based on the equation of motion and stress–strain relations for the real-valued time-dependent wavefields. The second approach is based on the equation of motion and stress–strain relation for the complex-valued time-harmonic wavefields. The second approach is applicable under unrestricted viscoelasticity and anisotropy. In both approaches, the energy–flux vector and other energy-related quantities are expressed as the sums of two contributions, namely the time-averaged and time-dependent constituents. The comparison of both approaches shows that they are fully replaceable for the Kelvin–Voigt viscoelastic model. This indicates that the second approach, which is considerably more general, is a better alternative for applications.

The energy-balance equations for viscoelastic media are complex-valued, which means that the energy-related quantities involved in them must satisfy two conditions, real and imaginary parts of an energy-balance equation. This is in contrast with perfectly elastic media, where a single, real-valued energy-balance equation must be satisfied.

For high-frequency wavefields, only the time-averaged energy contributions are often considered. In certain applications, however, it is useful to consider also the time-dependent contributions. The time-dependent energy–flux vector is known under different names (instantaneous Deschamp's pseudo-velocity vector, or pseudo-energy flux), and was investigated as an independent special entity (Červený 2001b). It is, however, shown in this paper that the time-dependent energy–flux vector is merely one of the two constituents of the complete energy–flux vector, and does not exist independently. Only the superposition of the time-averaged and time-dependent energy–flux vectors gives the complete energy–flux vector.

The energy-balance equations for individual waves are derived and discussed, both for the real- and complex-valued energy–flux vector. They show the relation of the energy–flux vector to other energy-related quantities, as the kinetic energy, strain energy, dissipated energy and energy supplied by internal sources. These energy-balance equations, however, are valid only for individual non-interfering waves. It would be useful to investigate the energy-related quantities also for trains of waves. Trains of waves can be generated by several sources (Mann *et al.* 1987), in the vicinity of a structural interface by waves generated by incidence of a wave at the interface, etc. Mann *et al.* (1987) investigated the energy flux of trains of waves in non-dissipative homogeneous fluids. They found that, even in this simple case, the energy–velocity field may display vortices in certain regions. For anisotropic viscoelastic media, this problem is still open to further research.

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