A NOTE ON DYNAMIC RAY TRACING IN RAY-CENTERED COORDINATES IN ANISOTROPIC INHOMOGENEOUS MEDIA

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ABSTRACT

Dynamic ray tracing plays an important role in paraxial ray methods. In this paper, dynamic ray tracing systems for inhomogeneous anisotropic media, consisting of four linear ordinary differential equations of the first order along the reference ray, are studied. The main attention is devoted to systems expressed in a particularly simple choice of ray-centered coordinates, here referred to as the standard ray-centered coordinates, and in wavefront orthonormal coordinates. These two systems, known from the literature, were derived independently and were given in different forms. In this paper it is proved that both systems are fully equivalent. Consequently, the dynamic ray tracing system, consisting of four equations in wavefront orthonormal coordinates, can also be used if we work in ray-centered coordinates, and vice versa.

Key words: dynamic ray tracing, inhomogeneous anisotropic media, ray-centered coordinates, wavefront orthonormal coordinates

1. INTRODUCTION

In three-dimensional laterally varying, isotropic or anisotropic media, dynamic ray tracing consists in the solution of four or six linear ordinary differential equations of the first order along an arbitrarily chosen reference ray Ω. The results of dynamic ray tracing have found broad application in paraxial ray methods, connected with the reference ray. They can also be used to calculate geometrical spreading, the curvature of the wavefront, the DRT propagator matrices (also called ray-propagator matrices), the travel times and slowness vectors in the vicinity of the reference ray, etc. They also play an important role in the solution of various boundary-value problems of two- or four-parametric systems of rays, and in certain extensions of the ray method.

In inhomogeneous anisotropic media, three coordinate systems have mostly been used to construct the dynamic ray tracing systems along the reference ray:

a) The first is the global Cartesian coordinate system x_i. The dynamic ray tracing system in Cartesian coordinates consists of six equations. In global Cartesian coordinates, however, the number of equations in the system cannot be reduced from six to four. See, for example, *Červený (1972)*, *Gajewski and Pšenčík (1987, 1990)*, *Norris (1987)*, *Farra (1989)*, *Kendall and Thomson (1989)*, *Kendall et al. (1992)*, *Červený (2001, sec. 4.14.1)*, *Chapman (2004)*, *Moser (2004)*, *Iversen (2004a,b)*, and *Moser and Červený (2007)*, where many other references can be found.

b) The second useful coordinate system is the ray-centered coordinate system q_i, connected with an arbitrarily taken reference ray. The main properties of the ray-centered coordinate system are: i) Reference ray Ω represents the q_3 -axis of the coordinate system, ii) The coordinate system is, in general, curvilinear, nonorthogonal. Coordinate axes q_1, q_2 may be chosen in various ways, see *Hanyga (1982)*, *Kendall et al. (1992)*, *Klimeš (1994, 2006)*, *Červený (2001, sec. 4.2.4/3)*, and *Červený and Moser (2007)*. For a particularly detailed treatment and for the comparison of several approaches see *Klimeš (2006)*. In general, the dynamic ray tracing system in ray-centered coordinates q_i consists of six equations. However, it can be simply reduced to four equations only, as the two remaining differential equations can be solved analytically (*Klimeš, 1994*).

   A very suitable option of ray-centered coordinates q_i is to take q_1, q_2 as Cartesian coordinates in a plane tangent to the wavefront at point *O*, at which the wavefront intersects reference ray Ω. We take point *O* to be the origin of the Cartesian coordinates, and specify two mutually perpendicular unit basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ tangent to the wavefront, varying along the ray in such a way that $\mathrm{d}\mathbf{e}_I/\mathrm{d}\tau$ is parallel to the local slowness vector $\mathbf{p}$ at any point of the ray. Here τ is the monotonic variable along the ray, corresponding to the travel time. We shall call this option the *standard ray-centered coordinate system* here, and use it systematically in the whole paper.

c) Wavefront orthonormal coordinates y_i have also been successfully used in deriving the dynamic ray tracing system for anisotropic inhomogeneous media, consisting of four equations. The wavefront orthonormal coordinate system is a local Cartesian coordinate system, the origin *O* of which moves with the wavefront along the reference ray Ω. The two coordinates y_1, y_2 of the wavefront orthonormal coordinate system are specified exactly in the same way as the standard ray-centered coordinates q_1, q_2. Consequently, the relevant basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ are mutually perpendicular unit vectors situated in the plane tangent to the wavefront at the reference ray, and $\mathrm{d}\mathbf{e}_I/\mathrm{d}\tau$ vary along the ray in the same way as in standard ray-centered coordinates. The third coordinate axis, however, is different from that in standard ray-centered coordinates. It is perpendicular to the wavefront at the reference ray, not tangent to the ray. Consequently, in anisotropic media it is parallel to the local slowness vector $\mathbf{p}$, and the reference ray is not a coordinate axis of the system. For isotropic media, see Section 6. The reader is reminded that the coordinate system y_i is locally Cartesian.

It is obvious that the dynamic ray tracing system, *consisting of six equations* in any selected local wavefront orthonormal coordinates y_i, is of no importance to us. Moreover, the orientation of the y_3-axis plays no role in the construction of the dynamic ray tracing system consisting of four equations. Hence, it is natural to try to derive the dynamic ray tracing system consisting of four equations in wavefront orthonormal coordinates y_1, y_2 directly. The derivation from the global Cartesian coordinates x_i is straightforward, as both coordinate systems are Cartesian. For a detailed derivation, see *Červený (1995, 2001, Sec. 4.2.2)*, and for a more concise treatment see *Bakker (1996)*. The derivation does not require the use of the formalism of curvilinear non-orthogonal coordinates with covariant and contravariant basis vectors.

Dynamic ray tracing systems consisting of four equations, in standard ray-centered coordinates and in wavefront orthonormal coordinates, should be the same as q_1, q_2 are the same as y_1, y_2. Both these systems, however, were derived independently and by different methods, and were given in different forms. For example, *Červený (2001)* derived and discussed both these systems, but did not prove that they were equivalent.

The main purpose of this paper is to prove that both systems, consisting of four equations, are fully equivalent, so that the dynamic ray tracing in wavefront orthonormal coordinates can be alternatively used if standard ray-centered coordinates are considered, and vice versa.

Section 2 introduces the ray tracing system in Hamiltonian form, which is equally valid for isotropic and anisotropic media. Section 3 is devoted to dynamic ray tracing in standard ray-centered coordinates, and Section 4 to dynamic ray tracing in wavefront orthonormal coordinates. Finally, Section 5 compares both dynamic ray tracing systems consisting of four equations and proves that they are fully equivalent. In Section 6, the derived dynamic ray tracing systems are simplified for isotropic media, and Section 7 offers some concluding remarks.

The vectors and matrices are denoted by bold symbols. In componental notations, the lower case indices $i, j, k, ...$ take the values 1,2,3, and the upper-case indices $I, J, K, ...$ the values 1,2. The Einstein summation convention for repeating indices is used.

2. RAY TRACING SYSTEM IN HAMILTONIAN FORM

We consider the eikonal equation for the travel-time field $T(\mathbf{x})$ in Hamiltonian form:

$$\mathcal{H}(\mathbf{x}, \mathbf{p}) = 0 \, . \tag{1}$$

Here $\mathcal{H}$ is the Hamiltonian, $\mathbf{x}$ the position vector, and $\mathbf{p}$ the slowness vector, $\mathbf{p} = \partial T / \partial \mathbf{x}$. We assume that the Hamiltonian is a homogeneous function of the second degree in p_i (with a possible additional constant). The kinematic ray tracing equations then read:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \boldsymbol{\mathcal{U}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \, , \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \boldsymbol{\eta} = -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} \quad . \tag{2}$$

Here τ is a monotonic variable along the ray, which represents the travel time. $\boldsymbol{\mathcal{U}}$ is the ray velocity vector (also referred to as the group velocity vector), tangent to the ray, and $\boldsymbol{\eta}$ measures the change of the slowness vector along the ray.

As Hamiltonian $\mathcal{H}$ is a homogeneous function of the second degree in p_i, it satisfies the relevant Euler equations for homogeneous functions, see *Červený (2001, eq. 2.2.24)*:

$$p_i \frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_j} = \frac{\partial \mathcal{H}}{\partial p_j} = \mathcal{U}_j \ , \qquad p_i \frac{\partial^2 \mathcal{H}}{\partial p_i \partial x_j} = 2 \frac{\partial \mathcal{H}}{\partial x_j} = -2\eta_j \tag{3}$$

Equations (3) will find useful applications in this paper.

We further denote

$$|\boldsymbol{\mathcal{U}}| = \mathcal{U} \ , \quad |\mathbf{p}| = 1/\mathcal{C} \ , \tag{4}$$

where $\mathcal{C}$ is the phase velocity, and $\mathcal{U}$ is the ray velocity (or group velocity).

3. DYNAMIC RAY TRACING IN RAY-CENTERED COORDINATES

The ray-centered coordinate system $\mathbf{q} = (q_1, q_2, q_3)$ connected with reference ray Ω, may be introduced in many ways, see *Klimeš (2006)*. The most important property of these coordinates is that reference ray Ω is the third coordinate axis of the system. Under q_3, we consider a monotonic variable τ along the ray, corresponding to the travel time. The coordinates q_1, q_2 are defined as Cartesian coordinates in the plane tangential to the wavefront, with the origin at the point of intersection *O* of the wavefront with the reference ray. More specifically, we shall introduce the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ of the local Cartesian coordinate system in the plane tangential to the wavefront as two mutually perpendicular unit vectors, satisfying the following ordinary differential equation of the first order along the reference ray:

$$\mathrm{d}\mathbf{e}_I/\mathrm{d}\tau = -(\mathbf{p}\cdot\mathbf{p})^{-1}(\mathbf{e}_I \cdot \boldsymbol{\eta})\mathbf{p} \ , \tag{5}$$

where $\boldsymbol{\eta}$ is given by Eq.(2). Eq.(5) guarantees that unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ represent mutually perpendicular unit vectors tangential to the moving wavefront along the whole reference ray, if they are mutually perpendicular, tangential to the wavefront and unit at any initial point of the ray (see *Červený, 2001*).

It would be possible to specify the ray-centered coordinates in many other ways. As mentioned in the Introduction, we shall refer to the version of ray-centered coordinates specified above as the *standard ray-centered coordinates*. In anisotropic media, the ray is not parallel to the slowness vector. Consequently, the standard ray-centered coordinate system is nonorthogonal. As the ray-centered coordinate system in anisotropic media is nonorthogonal, we can introduce two groups of basis vectors: the contravariant and covariant basis vectors. The contravariant basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \boldsymbol{\mathcal{U}}$ may be introduced in the following way: $\boldsymbol{\mathcal{U}}$ is the vector tangent to the ray, see Eq.(2), and $\mathbf{e}_1$, $\mathbf{e}_2$ are described above and satisfy Eq.(5). For $\tau = T$, $\boldsymbol{\mathcal{U}}$ is the ray-velocity vector. Similarly, the covariant basis vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3 = \mathbf{p}$ are introduced in the following way: $\mathbf{p}$ is the slowness vector, and $\mathbf{f}_1, \mathbf{f}_2$ are perpendicular to the ray. Both groups mutually satisfy the relation $\mathbf{f}_i \cdot \mathbf{e}_j = \delta_{ij}$.

The transformation matrices from the ray-centered coordinate system to the global Cartesian coordinate system and back are denoted by $H^{(q)}_{kl}$ and $\bar{H}^{(q)}_{kl}$, respectively, and read:

$$H_{kl}^{(q)} = \partial x_k / \partial q_l \ , \qquad \bar{H}_{ln}^{(q)} = \partial q_l / \partial x_n \ . \tag{6}$$

The 3×3 matrix $\bar{\mathbf{H}}^{(q)}$ is inverse to the 3×3 matrix $\mathbf{H}^{(q)}$,

$$\mathbf{H}^{(q)} \bar{\mathbf{H}}^{(q)} = \mathbf{I} \, , \tag{7}$$

but we denote it by a bar, as in *Červený (2001, sec. 4.2.4/3)*, to simplify certain componental equations. Matrix $\mathbf{H}^{(q)}$ is formed by three contravariant basis vectors $\mathbf{e}_i$, $\mathbf{H}^{(q)} \equiv (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and $\bar{\mathbf{H}}^{(q)}$ by three covariant basis vectors $\mathbf{f}_i$, $\bar{\mathbf{H}}^{(q)} \equiv (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)^T$. Note that

$$H_{i3}^{(q)} = \mathcal{U}_i \ , \quad \bar{H}_{3i}^{(q)} = p_i \ , \quad p_i H_{in}^{(q)} = \delta_{n3} \ , \quad \bar{H}_{ni}^{(q)} \mathcal{U}_i = \delta_{3n} \ . \tag{8}$$

We now introduce the paraxial quantities in the plane tangent to the wavefront:

$$Q_M^{(q)} = \partial q_M / \partial \gamma \ , \qquad P_M^{(q)} = \partial p_M^{(q)} / \partial \gamma \ , \tag{9}$$

where γ is the ray parameter and $p_M^{(q)} = \partial T / \partial q_M$. The partial derivatives with respect to γ are taken for constant τ and constant second ray parameter. Quantities $Q_M^{(q)}$ and $P_M^{(q)}$ can be determined along reference ray Ω by solving the dynamic ray tracing system in standard ray-centered coordinates consisting of four equations, assuming they are known at the initial point of the ray. The dynamic ray tracing equations in standard ray-centered coordinates consists of four equations and reads:

$$\begin{aligned} \mathrm{d}Q_M^{(q)} / \mathrm{d}\tau &= A_{MN}^{(q)} Q_N^{(q)} + B_{MN}^{(q)} P_N^{(q)} \ , \\ \mathrm{d}P_M^{(q)} / \mathrm{d}\tau &= -C_{MN}^{(q)} Q_N^{(q)} - D_{MN}^{(q)} P_N^{(q)} \ , \end{aligned} \tag{10}$$

where

$$\begin{aligned} A_{MN}^{(q)} &= \bar{H}_{Mn}^{(q)} H_{mN}^{(q)} \partial^2 \mathcal{H} / \partial p_n \partial x_m - \bar{H}_{Mi}^{(q)} \mathrm{d}H_{iN}^{(q)} / \mathrm{d}\tau \ , \\ B_{MN}^{(q)} &= \bar{H}_{Mn}^{(q)} \bar{H}_{Nm}^{(q)} \partial^2 \mathcal{H} / \partial p_n \partial p_m \ , \\ C_{MN}^{(q)} &= H_{nM}^{(q)} H_{mN}^{(q)} (\partial^2 \mathcal{H} / \partial x_n \partial x_m - \eta_n \eta_m) \, , \\ D_{MN}^{(q)} &= H_{nM}^{(q)} \bar{H}_{Nm}^{(q)} \partial^2 \mathcal{H} / \partial x_n \partial p_m - \bar{H}_{Ni}^{(q)} \mathrm{d}H_{iM}^{(q)} / \mathrm{d}\tau \ . \end{aligned} \tag{11}$$

See *Klimeš (1994, 2006)* and *Červený (2001, eqs. 4.2.77 and 4.2.78)*, where a detailed derivation can be found. The dynamic ray tracing system in standard ray-centered coordinates, consisting of six equations, is also derived and discussed in the latter reference.

Equations (11) can be simplified by removing derivatives $\mathrm{d}H_{iM}^{(q)} / \mathrm{d}\tau$ and $\mathrm{d}H_{iN}^{(q)} / \mathrm{d}\tau$. Using Eq.(5), we obtain

$$\mathrm{d}H_{iM}^{(q)} / \mathrm{d}\tau = -\mathcal{C}^2 (H_{mM}^{(q)} \eta_m) p_i \ . \tag{12}$$

The final equations for $A^{(q)}_{MN}$, $B^{(q)}_{MN}$, $C^{(q)}_{MN}$ and $D^{(q)}_{MN}$ then read

$$\begin{aligned} A^{(q)}_{MN} &= \bar{H}^{(q)}_{Mn} H^{(q)}_{mN} (\partial^2 \mathcal{H}/\partial p_n \partial x_m + \mathcal{C}^2 p_n \eta_m) \,, \\ B^{(q)}_{MN} &= \bar{H}^{(q)}_{Mn} \bar{H}^{(q)}_{Nm} \partial^2 \mathcal{H}/\partial p_n \partial p_m \,, \\ C^{(q)}_{MN} &= H^{(q)}_{nM} H^{(q)}_{mN} (\partial^2 \mathcal{H}/\partial x_n \partial x_m - \eta_n \eta_m) \,, \\ D^{(q)}_{MN} &= H^{(q)}_{nM} \bar{H}^{(q)}_{Nm} (\partial^2 \mathcal{H}/\partial x_n \partial p_m + \mathcal{C}^2 p_n \eta_m) \,. \end{aligned} \tag{13}$$

Note that Eqs.(13) are presented, in a slightly different form, also in *Červený et al. (2007).*

4. DYNAMIC RAY TRACING IN WAVEFRONT ORTHONORMAL COORDINATES

We introduce the wavefront orthonormal coordinates very briefly here. For more detailed information, see *Bakker (1996), Červený (1995, 2001, sec. 4.2.2).*

The wavefront orthonormal coordinates $\mathbf{y} = (y_1, y_2, y_3)$ are local Cartesian coordinates, with origin *O* moving with the wavefront along reference ray Ω. The y_1- and y_2-axes are introduced exactly in the same way as in the standard ray-centered coordinate system. The y_3-axis is, however, perpendicular to the wavefront at *O*, so that it is oriented along the slowness vector at *O*.

The 3×3 transformation matrices from the wavefront orthonormal coordinate system y_k at the specified point *O* to the global Cartesian coordinate system x_l read:

$$H^{(y)}_{kl} = \partial x_k / \partial y_l = \partial y_l / \partial x_k \,. \tag{14}$$

The 3×3 matrix $\mathbf{H}^{(y)}$ is unitary and is related to basis vectors $\mathbf{e}_i$ as follows:

$$H^{(y)}_{iK} = e_{Ki} \;, \qquad H^{(y)}_{i3} = e_{3i} = \mathcal{C} p_i \;, \tag{15}$$

where e_{ki} denotes the i-th Cartesian coordinate of $\mathbf{e}_k$. It satisfies the relation

$$\mathbf{H}^{(y)} \mathbf{H}^{(y)T} = \mathbf{I} \,. \tag{16}$$

In componental form, Eq.(16) reads $H^{(y)}_{nm} H^{(y)}_{im} = H^{(y)}_{nM} H^{(y)}_{iM} + H^{(y)}_{n3} H^{(y)}_{i3} = \delta_{in}$, which yields an important relation that we shall use in the following:

$$H^{(y)}_{nM} H^{(y)}_{iM} = \delta_{ni} - \mathcal{C}^2 p_n p_i \,. \tag{17}$$

We now introduce the paraxial quantities in the plane tangent to the wavefront:

$$Q^{(y)}_M = \partial y_M / \partial \gamma \;, \qquad P^{(y)}_M = \partial p^{(y)}_M / \partial \gamma \;, \tag{18}$$

where γ is a ray parameter and $p^{(y)}_M = \partial T / \partial y_M$. The partial derivatives with respect to γ are taken for constant τ. It is not difficult to show that $Q^{(y)}_M = Q^{(q)}_M$ and $P^{(y)}_M = P^{(q)}_M$ at any point of the ray, where $Q_M^{(q)}$ and $P_M^{(q)}$ are given by Eq.(9). We only have to realize that q_M equal y_M and $p_M^{(q)} = p_M^{(y)}$. Quantities $Q_M^{(y)}$ and $P_M^{(y)}$ can be computed along reference ray Ω by solving the dynamic ray tracing system in wavefront orthonormal coordinates, consisting of four equations, assuming they are known at the initial point of the ray. The dynamic ray tracing system in wavefront orthonormal coordinates, consisting of four equations reads:

$$\mathrm{d}Q_M^{(y)}/\mathrm{d}\tau = A_{MN}^{(y)}Q_N^{(y)} + B_{MN}^{(y)}P_N^{(y)}\,,$$
$$\mathrm{d}P_M^{(y)}/\mathrm{d}\tau = -C_{MN}^{(y)}Q_N^{(y)} - D_{MN}^{(y)}P_N^{(y)}\,. \qquad (19)$$

Here

$$A_{MN}^{(y)} = H_{iM}^{(y)}H_{jN}^{(y)}[\partial^2\mathcal{H}/\partial p_i\partial x_j + \mathcal{U}_i\eta_j]\,,$$
$$B_{MN}^{(y)} = H_{iM}^{(y)}H_{jN}^{(y)}[\partial^2\mathcal{H}/\partial p_i\partial p_j - \mathcal{U}_i\mathcal{U}_j]\,,$$
$$C_{MN}^{(y)} = H_{iM}^{(y)}H_{jN}^{(y)}[\partial^2\mathcal{H}/\partial x_i\partial x_j - \eta_i\eta_j]\,, \qquad (20)$$
$$D_{MN}^{(y)} = H_{iM}^{(y)}H_{jN}^{(y)}[\partial^2\mathcal{H}/\partial x_i\partial p_j + \eta_i\mathcal{U}_j]\,.$$

See *Červený (1995, eq. 4.14.19), Bakker (1996, eqs. 6 and 7),* and *Červený (2001, eqs. 4.2.31 and 4.2.32),* where a detailed derivation can be found.

5. COMPARISON OF DYNAMIC RAY TRACING SYSTEMS IN STANDARD RAY-CENTERED AND WAVEFRONT ORTHONORMAL COORDINATES

The dynamic ray tracing systems consisting of four equations in wavefront orthonormal coordinates, Eq.(19) with Eq.(20), and in standard ray-centered coordinates, Eq.(10) with Eq.(13), are expressed in different forms. In this section, we will compare them in greater detail. As $H_{iM}^{(q)}$ and $H_{iM}^{(y)}$ represent basis vectors $\mathbf{e}_M$, $M = 1,2$, in both systems, we will denote them simply H_{iM} (without the superscript).

We start with the dynamic ray tracing equations (19) in wavefront orthonormal coordinates, particularly with Eq.(20) for $A_{MN}^{(y)}$, $B_{MN}^{(y)}$, $C_{MN}^{(y)}$ and $D_{MN}^{(y)}$. Multiplying the first equation of (20) by H_{nM}, the second by $H_{nM}H_{mN}$, and the fourth equation by H_{mM}, we obtain:

$$H_{nM}A^{(y)}_{MN} = H_{nM}H_{iM}H_{jN}\left[\frac{\partial^2\mathcal{H}}{\partial p_i \partial x_j} + \mathcal{U}_i\eta_j\right] ,$$
$$H_{nM}H_{mN}B^{(y)}_{MN} = H_{nM}H_{mN}H_{iM}H_{jN}\left[\frac{\partial^2\mathcal{H}}{\partial p_i \partial p_j} - \mathcal{U}_i\mathcal{U}_j\right] ,$$
$$C^{(y)}_{MN} = H_{iM}H_{jN}\left[\frac{\partial^2\mathcal{H}}{\partial x_i \partial x_j} - \eta_i\eta_j\right] , \tag{21}$$
$$H_{mN}D^{(y)}_{MN} = H_{mN}H_{iM}H_{jN}\left[\frac{\partial^2\mathcal{H}}{\partial x_i \partial p_j} + \eta_i\mathcal{U}_j\right] .$$

Inserting relation (17) yields:

$$H_{nM}A^{(y)}_{MN} = (\delta_{ni} - \mathcal{C}^2 p_n p_i)H_{jN}\left[\partial^2\mathcal{H}/\partial p_i \partial x_j + \mathcal{U}_i\eta_j\right] ,$$
$$H_{nM}H_{mN}B^{(y)}_{MN} = (\delta_{ni} - \mathcal{C}^2 p_n p_i)(\delta_{mj} - \mathcal{C}^2 p_m p_j)\left[\partial^2\mathcal{H}/\partial p_i \partial p_j - \mathcal{U}_i\mathcal{U}_j\right],$$
$$C^{(y)}_{MN} = H_{iM}H_{jN}\left(\partial^2\mathcal{H}/\partial x_i \partial x_j - \eta_i\eta_j\right), \tag{22}$$
$$H_{mN}D^{(y)}_{MN} = (\delta_{jm} - \mathcal{C}^2 p_j p_m)H_{iM}\left(\partial^2\mathcal{H}/\partial x_i \partial p_j + \eta_i\mathcal{U}_j\right).$$

It is not difficult to calculate the r.h.s.'s of Eq.(22). If we take into account relations (3) and relation $p_i\mathcal{U}_i = 1$, we obtain:

$$H_{nM}A^{(y)}_{MN} = H_{jN}(\partial^2\mathcal{H}/\partial p_n \partial x_j + \mathcal{U}_n\eta_j + \mathcal{C}^2 p_n\eta_j),$$
$$H_{nM}H_{mN}B^{(y)}_{MN} = \partial^2\mathcal{H}/\partial p_n \partial p_m - \mathcal{U}_n\mathcal{U}_m ,$$
$$C^{(y)}_{MN} = H_{iM}H_{jN}(\partial^2\mathcal{H}/\partial x_i \partial x_j - \eta_i\eta_j), \tag{23}$$
$$H_{mN}D^{(y)}_{MN} = H_{iM}(\partial^2\mathcal{H}/\partial p_m \partial x_i + \eta_i\mathcal{U}_m + \mathcal{C}^2 p_m\eta_i).$$

We now multiply the first equation of (23) by $\bar{H}^{(q)}_{Kn}$, the second by $\bar{H}^{(q)}_{Kn}\bar{H}^{(q)}_{Lm}$, and the fourth by $\bar{H}^{(q)}_{Lm}$. In the first equation, we then use:

$$\bar{H}^{(q)}_{Kn}H_{nM} = \bar{H}^{(q)}_{Kn}H^{(q)}_{nM} = \delta_{KM} , \quad \bar{H}^{(q)}_{Kn}\mathcal{U}_n = 0 .$$

Analogous relations are also used in the second and fourth equations. Finally, we obtain:

$$A^{(y)}_{KN} = \bar{H}^{(q)}_{Ki}H^{(q)}_{jN}(\partial^2\mathcal{H}/\partial p_i \partial x_j + \mathcal{C}^2 p_i\eta_j),$$
$$B^{(y)}_{KN} = \bar{H}^{(q)}_{Ki}\bar{H}^{(q)}_{Nj}\partial^2\mathcal{H}/\partial p_i \partial p_j ,$$
$$C^{(y)}_{KN} = H^{(q)}_{iK}H^{(q)}_{jN}(\partial^2\mathcal{H}/\partial x_i \partial x_j - \eta_i\eta_j), \tag{24}$$
$$D^{(y)}_{KN} = H^{(q)}_{iK}\bar{H}^{(q)}_{Nj}(\partial^2\mathcal{H}/\partial x_i \partial p_j + \mathcal{C}^2 p_j\eta_i).$$

But this is exactly the same as in the dynamic ray tracing system in standard ray-centered coordinates, see Eq.(13).

Thus, the dynamic ray tracing systems consisting of four equations in standard ray-centered coordinates (10) with (13) and in wavefront orthonormal coordinates (19) and (20) differ only formally; actually both systems are equal. We may choose either.

6. ISOTROPIC MEDIA

In isotropic media, the dynamic ray tracing systems consisting of four equations simplify considerably, both in standard ray-centered and wavefront orthonormal coordinates.

We consider Hamiltonian $\mathcal{H}(x_i, p_i) = 1/2\ V^2 p_i p_i$, corresponding to the monotonic variable $\tau = T$ along the reference ray, where T is the travel time, and V is the propagation velocity ($V = \alpha$ for P waves, $V = \beta$ for S waves). Then,

$$\frac{\partial^2 \mathcal{H}}{\partial p_n \partial p_m} = V^2 \delta_{mn}\ , \quad \frac{\partial^2 \mathcal{H}}{\partial x_n \partial x_m} = \frac{1}{2V^2} \frac{\partial^2 V^2}{\partial x_n \partial x_m}\ ,$$
$$\frac{\partial^2 \mathcal{H}}{\partial p_n \partial x_m} = p_n \frac{\partial V^2}{\partial x_m}\ , \quad \frac{\partial^2 \mathcal{H}}{\partial x_n \partial p_m} = p_m \frac{\partial V^2}{\partial x_n}\ . \tag{25}$$

As basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ (with components $H^{(y)}_{i1}$ and $H^{(y)}_{i2}$, respectively) are perpendicular to slowness vector $\mathbf{p}$, we immediately see that

$$A^{(y)}_{MN} = 0\ , \qquad D^{(y)}_{MN} = 0\ . \tag{26}$$

As $\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually perpendicular and unit, we obtain:

$$B^{(y)}_{MN} = V^2 \delta_{MN}\ . \tag{27}$$

Finally, $C^{(y)}_{MN}$ is given by the relation:

$$\begin{aligned} C^{(y)}_{MN} &= H^{(y)}_{iM} H^{(y)}_{jN} \left[\frac{1}{2V^2} \frac{\partial^2 V^2}{\partial x_i \partial x_j} - \eta_i \eta_j \right] \\ &= H^{(y)}_{iM} H^{(y)}_{jN} \left[\frac{1}{V^2} \frac{\partial}{\partial x_i} \left(V \frac{\partial V}{\partial x_j} \right) - \eta_i \eta_j \right] \\ &= H^{(y)}_{iM} H^{(y)}_{jN} \left[\frac{1}{V} \frac{\partial^2 V}{\partial x_i \partial x_j} + \frac{1}{V^2} \frac{\partial V}{\partial x_i} \frac{\partial V}{\partial x_j} - \eta_i \eta_j \right] = \frac{1}{V} V_{MN}\ . \end{aligned} \tag{28}$$

Here we have used expression (2) for η_i, $\eta_i = -V^{-1} \partial V / \partial x_i$, and put

$$V_{MN} = H^{(y)}_{iM} H^{(y)}_{jN} \frac{\partial^2 V}{\partial x_i \partial x_j}\ . \tag{29}$$

Alternatively, we can express V_{MN} as follows: $V_{MN} = \partial^2 V/\partial y_M \partial y_N$. Inserting Eqs.(26)–(28) into Eq.(19), we obtain the final form of the dynamic ray tracing system in isotropic media:

$$\mathrm{d}Q^{(y)}_{MN}/\mathrm{d}\tau = V^2 P^{(y)}_{MN} \; , \qquad \mathrm{d}P^{(y)}_{MN}/\mathrm{d}\tau = -V^{-1} V_{MK} Q^{(y)}_{KN} \; . \tag{30}$$

Exactly the same system was derived in *Červený (2001, eq. 4.1.65)*. If the chosen variable along the ray is the arclength, system (30) reads:

$$\mathrm{d}Q^{(y)}_{MN}/\mathrm{d}s = V P^{(y)}_{MN} \; , \quad \mathrm{d}P^{(y)}_{MN}/\mathrm{d}s = -V^{-2} V_{MK} Q^{(y)}_{KM} \; . \tag{31}$$

System (31) was first introduced to seismology by *Popov and Pšenčík (1978a,b)*. Many alternative forms of Eq.(31) are possible, see *Červený (2001, Sec. 4.1)*.

Analogously, we can show that the dynamic ray tracing systems (10) with (13) in standard ray-centered coordinates also leads to Eqs.(30) and (31) in isotropic media.

7. CONCLUDING REMARKS

As we have proved, the dynamic ray tracing systems, consisting of four equations, in standard ray-centered coordinates and in wavefront orthonormal coordinates are fully equivalent. Consequently, both dynamic ray tracing systems can be used alternatively.

The numerical efficiency of both systems is practically the same, as most of computer time in computations is spent on evaluating the second derivatives of the Hamiltonian at any point of the ray. The dynamic ray tracing system in wavefront orthonormal coordinates seems to be slightly simpler, as it does not require computation of the covariant basis vectors $\mathbf{f}_1$, $\mathbf{f}_2$. Moreover, the dynamic ray tracing system in wavefront orthonormal coordinates is more transparent, as it does not require the covariant and contravariant basis vectors to be distinguished; the mutually perpendicular unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, tangent to the wavefront, are quite sufficient.

As the dynamic ray tracing system is linear, it is possible to construct the propagator matrix corresponding to the system along the reference ray. It is usually referred to as the DRT propagator matrix or ray-propagator matrix. It represents the fundamental matrix of the system, which is equal to the identity matrix at some initial point. It has found broad applications in paraxial ray methods. Clearly, the 4×4 DRT propagator matrices in both the standard ray tracing coordinates and in wavefront orthonormal coordinates are fully alternative.

As the 4×4 DRT propagator matrix in standard ray-centered coordinates can also be used to determine the 6×6 DRT propagator matrix in ray-centered coordinates and in Cartesian coordinates (see *Červený and Moser, 2007*), the same applies to the 4×4 DRT propagator matrix in wavefront orthonormal coordinates. They are sufficient to determine the full 6×6 DRT propagator matrices in standard ray-centered and in Cartesian coordinates. Only some simple transformation matrices must be computed and applied at the initial and end points of the ray (not along the whole ray).

It is well known that the DRT propagator matrices can be chained at arbitrary points of the ray. This is particularly important in anisotropic inhomogeneous media including *smoothly curved structural interfaces*. The so-called interface propagator matrices must be included in the chain wherever the ray strikes the interface. The interface propagator matrices for wavefront orthonormal coordinates are derived in *Červený (2001, sec. 4.14.4)*, and in a more compact form in *Červený and Moser (2007)*. Consequently, all conclusions of this paper remain valid even for layered anisotropic inhomogeneous media.

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