REFLECTION/TRANSMISSION LAWS FOR SLOWNESS VECTORS IN VISCOELASTIC ANISOTROPIC MEDIA

V. ČERVENÝ

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic (vcerveny@seis.karlov.mff.cuni.cz)

Received: November 24, 2006; Revised: February 23, 2007; Accepted: February 23, 2007

ABSTRACT

The reflection/transmission laws (R/T laws) of plane waves at a plane interface between two homogeneous anisotropic viscoelastic (dissipative) halfspaces are discussed. Algorithms for determining the slowness vectors of reflected/transmitted plane waves from the known slowness vector of the incident wave are proposed. In viscoelastic media, the slowness vectors of plane waves are complex-valued, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$*, where* $\mathbf{P}$ *is the propagation vector, and* $\mathbf{A}$ *the attenuation vector. The proposed algorithms may be applied to bulk plane waves (*$\mathbf{A} = \mathbf{0}$*), homogeneous plane waves (*$\mathbf{A} \neq \mathbf{0}$*,* $\mathbf{P}$ *and* $\mathbf{A}$ *parallel), and inhomogeneous plane waves (*$\mathbf{A} \neq \mathbf{0}$*,* $\mathbf{P}$ *and* $\mathbf{A}$ *non-parallel). The manner, in which the slowness vector is specified, plays an important role in the algorithms. For unrestricted anisotropy and viscoelasticity, the algorithms require an algebraic equation of the sixth degree to be solved in each halfspace. The degree of the algebraic equation decreases to four or two for simpler cases (isotropic media, plane waves in symmetry planes of anisotropic media). The physical consequences of the proposed algorithms are discussed in detail.*

Key words: viscoelastic anisotropic media, reflection and transmission laws, Snell's law

1. INTRODUCTION

Consider a planar structural interface Σ between two homogeneous anisotropic viscoelastic halfspaces, and a time-harmonic plane wave incident from one halfspace at it. Denote by $\mathbf{p}^{inc}$ the slowness vector of the incident wave. This paper deals with the determination of slowness vectors $\mathbf{p}^r$ of any reflected/transmitted waves from known $\mathbf{p}^{inc}$. Both $\mathbf{p}^r$ and $\mathbf{p}^{inc}$may be complex-valued.

For an interface between two perfectly elastic, isotropic media, the analytical solutions of the above formulated problem are well-known. It will be instructive to recapitulate briefly the well-known solutions for perfectly elastic isotropic media, and only then to treat the viscoelastic anisotropic media.

© 2007 StudiaGeo s.r.o., Prague

We denote by $\mathbf{n}$ the unit vector normal to interface Σ, oriented to any side of it. Denote also by i^{inc} the angle of incidence, which specifies the acute angle between $\mathbf{n}$ and the real-valued slowness vector of the incident wave $\mathbf{p}^{inc}$. Snell's law, in its classical form, relates the angle i^r of any reflected/transmitted wave to the angle of incidence i^{inc}. It reads

$$\sin i^{inc}/V^{inc} = \sin i^r/V^r. \tag{1}$$

The reflection/transmission (R/T) angle i^r is again defined as the acute angle between $\mathbf{n}$ and the slowness vector $\mathbf{p}^r$ of the R/T wave. Symbols V^{inc} and V^r denote the real-valued velocities of the incident wave and of the selected R/T wave, respectively. They may correspond to P wave velocities α, or S wave velocities β, depending on the modes of the incident and R/T waves being considered. For monotypic reflected waves $(\mathrm{P} \rightarrow \mathrm{P}$, $\mathrm{S} \rightarrow \mathrm{S})$, $V^{inc} = V^r$, and, consequently, $i^r = i^{inc}$.

Equation (1) is valid for any incident (P or S), reflected (P or S) and transmitted (P or S) waves. It actually expresses the fact that the tangential component of the slowness vector of any R/T wave generated at interface Σ is the same as the tangential component of the slowness vector of the incident wave at Σ. This is the well-known condition which must be satisfied at any structural interface.

For given i^{inc}, V^{inc} and V^r, angle i^r may be complex-valued. This occurs if $\sin i^{inc} > V^{inc}/V^r$ (postcritical incidence). The R/T plane wave corresponding to velocity V^r is then inhomogeneous, and its amplitudes decay exponentially in the direction perpendicular to Σ, away from the interface.

Snell's law is one of the basic laws in the seismic ray method in inhomogeneous, isotropic, layered media. Nevertheless, it may be very useful to consider it in a more general form, formulated at the beginning of this section, in which the slowness vector $\mathbf{p}^r$ of any R/T wave is to be computed from the slowness vector $\mathbf{p}^{inc}$ of incident wave. For isotropic perfectly elastic halfspaces, it is easy to derive the solution, both analytically and geometrically. It reads:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\left[(\mathbf{p}^{inc}\cdot\mathbf{n}) \mp \sqrt{(1/V^r)^2 - (1/V^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2}\right] . \tag{2}$$

For $\mathbf{n}$ oriented to the side where the transmitted wave propagates, the upper sign in Eq.(2) corresponds to transmitted waves, and the lower sign (+) to reflected waves. For the uncorverted reflected waves, Eq.(2) simplifies to $\mathbf{p}^r = \mathbf{p}^{inc} - 2\mathbf{n}(\mathbf{p}^{inc}\cdot\mathbf{n})$. Eq.(2) remains valid even for inhomogeneous R/T waves. Then the square root in Eq.(2) is purely imaginary, with the sign chosen in such a way to obtain the R/T wave whose amplitude decays exponentially from the interface. Eq.(2) has been broadly used in the seismic ray method of laterally varying layered structures, see *Červený et al. (1977, Eq.(3.7))*, *Červený (2001, Eq.(2.3.29))*, *Robein (2003, Eq.(1.34))*, etc., and is included in most of the 2-D and 3-D ray tracing computer packages for isotropic, laterally varying, layered structures. Eq.(2) can be applied directly to computing the slowness vectors of R/T waves at an arbitrarily oriented curved interface, from an arbitrarily oriented slowness vector of the incident wave. It is expressed in a suitable vectorial, coordinate independent form. Contrary to Eq.(1), Eq.(2) yields the complete slowness vectors of R/T waves, including both their components: normal and tangential to Σ. Note that the cross-product of Eq.(2) with $\mathbf{n}$ yields $\mathbf{p}^r \times \mathbf{n} = \mathbf{p}^{inc} \times \mathbf{n}$. This equation indicates that the tangential components of the slowness vectors of the incident and all R/T waves are equal, and is analogous to Eq.(1). Equations for the normal components of the slowness vectors of R/T waves, contained in Eq.(2), follow simply from Eq.(2), taking the scalar product of Eq.(2) with $\mathbf{n}$. In Section 4, we shall prove that Eq.(2) remains valid even for isotropic viscoelastic media, both for homogeneous and inhomogeneous plane waves.

Let us now consider the viscoelastic anisotropic media. Formally, the expression for the slowness vector $\mathbf{p}^r$ of the R/T waves remains very similar to Eq.(2), only the square root in this expression must be replaced by a root of a complex-valued algebraic equation. For unrestricted anisotropy and viscoelasticity, the degree of the algebraic equation is six, see Section 4. The six roots correspond to individual P, S1 and S2 waves, propagating in the halfspace under consideration away from the interface and towards to it. In isotropic media, see Eq.(2), the number of waves is reduced to four, due to the degeneracy of S waves.

Thus, in general, we consider twelve waves, six in either halfspace. (The incident wave is one of them.) The tangential components of the slowness vectors of all these waves must be the same. Consequently, if we wish to study some general properties of the R/T laws, we can study them using these general equations. They remain valid even if we do not specify the type of incident wave and the halfspace, from which the incident wave is approaching; it is actually sufficient to consider only the tangential component of the slowness vector of this wave along the interface Σ.

Only if we wish to solve a specific case of a selected incident wave, the number of R/T waves must be reduced from twelve to six: three reflected and three transmitted. Such R/T waves must be selected, which propagate away from the interface. Analogously, the incident wave under consideration must propagate towards interface Σ.

Thus, a proper selection criterion must be used to choose the R/T waves propagating away from Σ, and the incident wave propagating towards Σ correctly. For isotropic perfectly elastic media and real-valued $\mathbf{p}^r$, the selection criterion is simple: For reflected waves, the signs of $\mathbf{p}^r \cdot \mathbf{n}$ and $\mathbf{p}^{inc} \cdot \mathbf{n}$ must be opposite, and for the transmitted waves, the signs of $\mathbf{p}^r \cdot \mathbf{n}$ and $\mathbf{p}^{inc} \cdot \mathbf{n}$ must be the same. For anisotropic perfectly elastic media and real-valued $\mathbf{p}^r$, analogous selection criteria are based on the directions of the time-averaged energy flux, not on the directions of the slowness vector. Actually, the criterion based on the direction of the time-averaged energy flux is universal, valid also for isotropic media, as the time-averaged energy flux is parallel to the slowness vector in isotropic media. For complex-valued $\mathbf{p}^r$ in perfectly elastic media (postcritical R/T waves), the selection criterion requires that the amplitudes of R/T waves decay exponentially in the direction away from the interface. See *Fedorov (1968)*, *Henneke (1972)*, *Gajewski and Pšenčík (1987)* and *Červený (2001, p. 51)*.

For viscoelastic media, however, the selection criteria are still a subject of research, see *Rund (2006)* and *Krebes and Daley (2007)*, where many other references can be found. In this paper, the selection criteria are not treated at all; we consider the slowness vectors of all 12 waves (P, S1 and S2 waves, propagating towards Σ and outwards from Σ), which have the same tangential components of slowness vectors as the incident wave.

2. PLANE WAVES IN VISCOELASTIC ANISOTROPIC MEDIA

Consider a time-harmonic plane wave propagating in a homogeneous viscoelastic anisotropic unbounded medium, specified by complex-valued density-normalised viscoelastic moduli a_{ijkl} $(i,j,k,l=1,2,3)$, satisfying the symmetry relations $a_{ijkl}=a_{jikl}=a_{ijlk}=a_{klij}$. Alternatively to a_{ijkl}, we also use the complex-valued density-normalised viscoelastic moduli $A_{\alpha\beta}$ $(\alpha,\beta=1,...,6)$ in the Voigt notation, and assume that the 6×6 matrix $\mathrm{Re}(A_{\alpha\beta})$ is positive definite, and the 6×6 matrix $-\mathrm{Im}(A_{\alpha\beta})$ positive definite, or zero.

The propagation of time-harmonic plane waves in viscoelastic anisotropic media was investigated in detail in *Červený and Pšenčík (2005)*. For this reason, we shall be brief here. We shall also use the same notations as in *Červený and Pšenčík (2005)*. Any time-harmonic plane wave is given by the relation

$$u_j(x_k,t)=U_j\exp[-\mathrm{i}\omega(t-p_nx_n)], \qquad (3)$$

where x_n are Cartesian coordinates, t the running time and ω a fixed real-valued positive circular frequency. Further, u_j, p_j and U_j are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, respectively. It follows from the elastodynamic equation that Eq.(3) represents a plane wave only if the following constraint relation is satisfied:

$$\det[a_{ijkl}p_jp_l-\delta_{ik}]=0. \qquad (4)$$

Slowness vector $\mathbf{p}$ is complex-valued,

$$\mathbf{p}=\mathbf{P}+\mathrm{i}\mathbf{A}, \qquad (5)$$

where the real-valued propagation vector $\mathbf{P}$ is perpendicular to the plane of constant phase, and is oriented in the direction of propagation of the wavefront, and the real-valued attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitude, and oriented in the direction of maximum decay of amplitudes.

The plane wave with $\mathbf{A}=\mathbf{0}$ (real-valued slowness vector) is referred to here as the bulk plane wave, the plane wave with parallel $\mathbf{P}$ and $\mathbf{A}(\neq\mathbf{0})$ as the homogeneous plane wave, and the plane wave with non-parallel $\mathbf{P}$ and $\mathbf{A}(\neq\mathbf{0})$ as the inhomogeneous plane wave.

Throughout this paper, we consider real-valued frequency ω. In recent studies of inhomogeneous plane waves, propagating in viscoelastic media, also complex-valued frequency ω has been often used. See, for example, *Deschamps et al. (1997)* and *Declercq et al. (2005)*, where many other references can be found.

Propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ cannot be chosen arbitrarily; they must satisfy the constraint relation (4). We now explain two specifications of the slowness vector, which will be useful in the discussion of the reflection/transmission laws: the componental specification and the mixed specification of the slowness vector.

In the **componental specification**, we consider a known reference plane Σ, with a real-valued unit vector $\mathbf{n}$ perpendicular to it, and along which the complex-valued tangential components $\mathbf{p}^{\Sigma}$ of the slowness vector are known. We wish to determine the unknown normal component $\sigma\mathbf{n}$ of the slowness vector, and, consequently, the complete slowness vector $\mathbf{p}$. Reference plane Σ may represent a structural interface, the surface of the Earth, a formal plane in a homogeneous medium, or the wavefront. The componental specification of slowness vector $\mathbf{p}$ then reads,

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma} \ , \qquad \text{with } \mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0 \ \ . \tag{6}$$

The unknown complex-valued quantity Σ must be determined by inserting Eq.(6) into the constraint relation (4). Consider a halfspace bounded by Σ, described by complex-valued viscoelastic moduli a_{ijkl}. Inserting Eq.(6) into Eq.(4) yields an equation for Σ:

$$\det[a_{ijkl}(\sigma n_j + p_j^{\Sigma})(\sigma n_l + p_l^{\Sigma}) - \delta_{ik}] = 0 \ \ . \tag{7}$$

Given a_{ijkl}, n_j and p_j^{Σ}, Eq.(7) represents a complex-valued algebraic equation of the sixth degree for Σ. Consequently, six complex-valued roots are obtained. They correspond to the P, S1 and S2 waves, propagating in the halfspace under consideration, towards Σ and away from Σ. All these six waves have the same tangential components $\mathbf{p}^{\Sigma}$ of the slowness vectors at reference plane Σ. The propagation directions of the six plane waves, however, are mutually different and are not known in advance in the componental specification.

The situation in the second halfspace bounded by Σ is quite analogous but appropriate a_{ijkl} must be used. Again, we obtain six roots, corresponding to the P, S1 and S2 waves, propagating in this halfspace, towards and away from Σ. All these waves again have the same tangential component $\mathbf{p}^{\Sigma}$ of the slowness vectors at Σ.

Once the six roots σ in the halfspace under consideration have been found, the relevant real-valued propagation vector $\mathbf{P}$, real-valued attenuation vector $\mathbf{A}$, unit propagation vector $\mathbf{N}$, unit attenuation vector $\mathbf{M}$, phase velocity $\mathcal{C}$ and attenuation angle γ ($\cos\gamma = \mathbf{N}\cdot\mathbf{M}$) can be determined from Eqs.(6) and (7):

$$\begin{gathered}
\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma + \mathrm{Re}\mathbf{p}^{\Sigma} \ , \quad \mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + \mathrm{Im}\mathbf{p}^{\Sigma} \ , \\
|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + \mathrm{Re}\mathbf{p}^{\Sigma}\,\mathrm{Re}\mathbf{p}^{\Sigma}]^{1/2} \ , \quad |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + \mathrm{Im}\mathbf{p}^{\Sigma}\,\mathrm{Im}\mathbf{p}^{\Sigma}]^{1/2} \ , \\
\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\
\cos\gamma = [\mathrm{Re}\mathbf{p}^{\Sigma}\,\mathrm{Im}\mathbf{p}^{\Sigma} + \mathrm{Re}\sigma\,\mathrm{Im}\sigma]/|\mathbf{P}||\mathbf{A}| \ , \quad \mathcal{C} = 1/|\mathbf{P}| \ .
\end{gathered} \tag{8}$$

The mixed specification is a special case of the componental specification, in which reference plane Σ is a wavefront (*T* = const.). As $\mathrm{Re}(\mathbf{p}^{\Sigma})$ vanishes along wavefront Σ, the unit propagation vectors $\mathbf{N}$ of all six waves equal $\pm\mathbf{n}$. Thus, all the six plane waves propagate in the direction of $\mathbf{n}$or $-\mathbf{n}$.

As $\mathbf{p}^{\boldsymbol{\Sigma}}$ is purely imaginary, we can introduce it by a simple relation:

$$\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m} \ , \qquad \text{with } \ \mathbf{m}\cdot\mathbf{N} \ = 0 \ . \tag{9}$$

Here $\mathbf{m}$ is a real-valued unit vector, situated in the wavefront (perpendicular to $\mathbf{N}$), and D is a scalar, real-valued quantity. The mixed specification of the slowness vector then reads

$$\mathbf{p} = \sigma\mathbf{N} + \mathrm{i}D\mathbf{m} \ , \qquad \text{with} \ \ \mathbf{m}\cdot\mathbf{n} = 0 \ . \tag{10}$$

The unknown complex-valued quantity σ must again be determined from the constraint relation (4):

$$\det[a_{ijkl}(\sigma N_j + \mathrm{i}Dm_j)(\sigma N_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \ . \tag{11}$$

Given a_{ijkl}, N_j, m_j and D, Eq.(11) again yields six complex-valued roots, which correspond to P, S1 and S2 waves, propagating in the direction of $\mathbf{N}$.

It is simple to see from Eq.(10) that the plane wave is homogeneous for $D = 0$, and inhomogeneous for $D \neq 0$. For this reason, we call D the *inhomogeneity parameter* and $|D|$ the *inhomogeneity strength*. The bulk plane wave is specified by $D = 0$ and $\mathrm{Im}\sigma = 0$. Any choice of mutually perpendicular vectors $\mathbf{n}$ and $\mathbf{m}$ specifies the propagation-attenuation plane $\Sigma^{\|}$, which contains both propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$.

For a given plane wave with selected $\mathbf{N}$, the propagation vector $\mathbf{P}$ and the attenuation vector $\mathbf{A}$ are expressed in terms of $\mathbf{N}$, $\mathbf{m}$, D and σ as follows, see Eq.(10):

$$\mathbf{P} = \mathbf{N}\,\mathrm{Re}\sigma \ , \qquad \mathbf{A} = \mathbf{N}\,\mathrm{Im}\sigma + D\mathbf{m} \ . \tag{12}$$

As $\mathbf{N}\times(\mathbf{m}\times\mathbf{N}) = \mathbf{m}$, the second part of Eq.(12) also yields:

$$D\mathbf{m} = \mathbf{N}\times(\mathbf{A}\times\mathbf{N}) \ , \tag{13}$$

and, consequently, for $D \neq 0$,

$$D = |\mathbf{N}\times(\mathbf{A}\times\mathbf{N})| \ , \quad \mathbf{m} = (\mathbf{N}\times(\mathbf{A}\times\mathbf{N}))/D \ . \tag{14}$$

Alternatively, we may take both D and $\mathbf{m}$ with opposite signs. We further obtain

$$|\mathbf{P}| = |\mathrm{Re}\sigma| \ , \ \ |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \ \ \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ ,$$
$$\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , \ \ \cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \tag{15}$$
$$\text{where} \ \ \epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1$$

3. R/T LAWS: A GENERAL VERSION

The purpose of this section is to determine the complex-valued slowness vector $\mathbf{p}^r = \mathbf{P}^r + \mathrm{i}\mathbf{A}^r$ of any R/T plane wave from the known complex-valued slowness vector $\mathbf{p}^{inc} = \mathbf{P}^{inc} + \mathrm{i}\mathbf{A}^{inc}$ of the incident plane wave. In this case, it is possible to use the *componental specification* (6) for the R/T wave, and to solve the relevant algebraic equation (7) for σ. The mixed specification of the slowness vector (10) is not required at all. For more general and practical approach, in which the slowness vector of the incident wave $\mathbf{P}^{inc}$ is specified by the unit vector perpendicular to the wavefront $\mathbf{N}^{inc}$, unit vector $\mathbf{m}^{inc}$ and the inhomogeneity parameter D^{inc}, see Section 5.

Again consider plane interface Σ, the normal unit vector $\mathbf{n}$ perpendicular to Σ, oriented to either side of Σ, and the slowness vector of incident wave $\mathbf{p}^{inc}$. We assume that $\mathbf{p}^{inc}$ satisfies the constraint relation (4), where a_{ijkl} correspond to the appropriate halfspace. Here we do not specify the slowness vector of the incident wave in more detail; the incident wave may propagate in any of the two halfspaces. We wish to determine the slowness vectors $\mathbf{p}^r$ of reflected/transmitted waves in the halfspace, described by the density-normalised viscoelastic moduli a^r_{ijkl}. The procedure in the second halfspace would be quite analogous.

Let us emphasize again that the superscript "r" is used for all reflected and transmitted waves throughout the paper. More specifically, it is not related to reflected waves only, but also to transmitted waves. For monotypically reflected waves (P $\to$ P, S $\to$ S), the presented expressions may be simplified.

First we determine $\mathbf{p}^\Sigma$, the component of $\mathbf{p}^{inc}$ into reference plane Σ:

$$\mathbf{p}^\Sigma = \mathbf{n} \times (\mathbf{p}^{inc} \times \mathbf{n}) = \mathbf{p}^{inc} - \mathbf{n}(\mathbf{p}^{inc} \cdot \mathbf{n}) \,. \tag{16}$$

If we use the notation

$$\mathbf{p}^\Sigma = \mathbf{P}^\Sigma + \mathrm{i}\mathbf{A}^\Sigma \ , \quad \mathbf{p}^{inc} = \mathbf{P}^{inc} + \mathrm{i}\mathbf{A}^{inc} \ , \tag{17}$$

we obtain

$$\mathbf{P}^\Sigma = \mathbf{n} \times (\mathbf{P}^{inc} \times \mathbf{n}) = \mathbf{P}^{inc} - \mathbf{n}(\mathbf{P}^{inc} \cdot \mathbf{n}) \ ,$$
$$\mathbf{A}^\Sigma = \mathbf{n} \times (\mathbf{A}^{inc} \times \mathbf{n}) = \mathbf{A}^{inc} - \mathbf{n}(\mathbf{A}^{inc} \cdot \mathbf{n}) \ . \tag{18}$$

Using Eqs.(6) and (7), we obtain the slowness vector $\mathbf{p}^r$ of any reflected/transmitted wave in terms of $\mathbf{p}^\Sigma$,

$$\mathbf{p}^r = \sigma^r \mathbf{n} + \mathbf{p}^\Sigma \ , \tag{19}$$

and in terms of $\mathbf{p}^{inc}$, see Eq.(16),

$$\mathbf{p}^r = \sigma^r \mathbf{n} + \mathbf{n} \times (\mathbf{p}^{inc} \times \mathbf{n}) = \mathbf{p}^{inc} - \mathbf{n}[\mathbf{p}^{inc} \cdot \mathbf{n} - \sigma^r] \ . \tag{20}$$

Here $\sigma^r = \sigma^r(a^r_{ijkl}, \mathbf{p}^{inc}, \mathbf{n})$ is any of the six roots of the algebraic equation (7), in which a^r_{ijkl} have been substituted for a_{ijkl},

$$\det[a^r_{ijkl}(\sigma^r n_j + p^\Sigma_j)(\sigma^r n_l + p^\Sigma_l) - \delta_{ik}] = 0 \ , \tag{21}$$

and $\mathbf{p}^\Sigma$ is given by Eq.(16).

We separate the real-valued and imaginary-valued parts of Eq.(19), and obtain:

$$\mathbf{P}^r = \mathbf{n}\mathrm{Re}\sigma^r + \mathbf{P}^\Sigma \ , \quad \mathbf{A}^r = \mathbf{n}\mathrm{Im}\sigma^r + \mathbf{A}^\Sigma \ . \tag{22}$$

Alternatively, in terms of $\mathbf{p}^{inc}$ and $\mathbf{A}^{inc}$:

$$\mathbf{P}^r = \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc} \cdot \mathbf{n}) - \mathrm{Re}\sigma^r] = \mathbf{n}\mathrm{Re}\sigma^r + \mathbf{n} \times (\mathbf{P}^{inc} \times \mathbf{n}) \ ,$$
$$\mathbf{A}^r = \mathbf{A}^{inc} - \mathbf{n}[(\mathbf{A}^{inc} \cdot \mathbf{n}) - \mathrm{Im}\sigma^r] = \mathbf{n}\mathrm{Im}\sigma^r + \mathbf{n} \times (\mathbf{A}^{inc} \times \mathbf{n}) \ . \tag{23}$$

Equations (19)–(20) and (22)–(23), with Eqs.(16)–(18) and (21) represent the final forms of the R/T laws. They are fully alternative and quite general. They are valid for isotropic and anisotropic, viscoelastic and perfectly elastic media. They are valid for bulk plane waves, homogeneous and inhomogeneous plane waves, for reflected and transmitted waves (we only specify properly the viscoelastic moduli a^r_{ijkl} in the halfspace under consideration), and for subcritically and postcritically R/T waves.

For different approaches to the R/T laws in viscoelastic anisotropic media see *Deschamps (1994), Caviglia and Morro (1999), Carcione (2001), Declercq et al. (2005),* etc.

Now define the plane of incidence Σ^{inc} by vectors $\mathbf{n}$ and $\mathbf{P}^{inc}$, and the attenuation plane of incidence Σ^{att} by $\mathbf{n}$ and $\mathbf{A}^{inc}$. For a general inhomogeneous incident wave, planes Σ^{inc} and Σ^{att} do not coincide. Certain general rules, however, are valid even in this case:

1. The propagation vectors of incident wave $\mathbf{P}^{inc}$ and of all R/T waves $\mathbf{P}^r$ are situated in the plane of incidence Σ^{inc}, see the first equation of (23).
2. The attenuation vectors of incident wave $\mathbf{A}^{inc}$ and of all R/T waves $\mathbf{A}^r$ are situated in the attenuation plane of incidence Σ^{att}, passing through the point of incidence, see the second equation of (23).
3. Taking the cross-product of Eq.(20) with $\mathbf{n}$ confirms that the tangential components of slowness vectors of all R/T waves are the same as the tangential component of the incident plane wave:

$$\mathbf{p}^r \times \mathbf{n} = \mathbf{p}^{inc} \times \mathbf{n} \ . \tag{24}$$

   This implies

$$\mathbf{P}^r \times \mathbf{n} = \mathbf{P}^{inc} \times \mathbf{n} \ , \quad \mathbf{A}^r \times \mathbf{n} = \mathbf{A}^{inc} \times \mathbf{n} \ . \tag{25}$$

   Thus, the tangential components of attenuation vector $\mathbf{A}$ of incident and generated waves are the same, similarly as the tangential components of the propagation vectors $\mathbf{P}$.
4. If we introduce the angle of incidence θ^{inc} and the R/T angle θ^r by relations

$$\sin\theta^{inc} = |\mathbf{N}^{inc} \times \mathbf{n}| \ , \quad \sin\theta^r = |\mathbf{N}^r \times \mathbf{n}| \ , \tag{26}$$

   where $\mathbf{N}^{inc}$ and $\mathbf{N}^r$ are unit vectors perpendicular to the wavefronts of the incident wave and of arbitrarily selected R/T wave, respectively, we obtain from the first equation of (25)

$$\sin\theta^{inc}/\mathcal{C}^{inc} = \sin\theta^r/\mathcal{C}^r \ . \tag{27}$$

   This form of Snell's law is valid quite generally; viscoelastic anisotropic media and inhomogeneous plane waves included. For bulk plane waves propagating in isotropic perfectly elastic media, it coincides with the classical version of Snell's law (as $\mathcal{C}^{inc} = V^{inc}$, $\mathcal{C}^r = V^r$, $\theta^{inc} = i^{inc}$ and $\theta^r = i^r$ in this case).

5. The second equation of (25) yields an analogous "Snell's law" for attenuation, $|\mathbf{A}^r|$ and $|\mathbf{A}^{inc}|$. If we introduce angles ζ^{inc} and ζ^r by relations:

$$\sin\zeta^{inc} = |\mathbf{M}^{inc} \times \mathbf{n}| \ , \quad \sin\zeta^r = |\mathbf{M}^r \times \mathbf{n}| \ , \tag{28}$$

we obtain

$$|\mathbf{A}^r| \sin\zeta^r = |\mathbf{A}^{inc}| \sin\zeta^{inc} \ . \tag{29}$$

6. Taking the scalar product of (20) and (23) with **n**, we obtain simple expressions for the normal components of the slowness vectors of all R/T waves:

$$\mathbf{p}^r \cdot \mathbf{n} = \sigma^r \ , \quad \mathbf{P}^r \cdot \mathbf{n} = \mathrm{Re}\sigma^r \ , \quad \mathbf{A}^r \cdot \mathbf{n} = \mathrm{Im}\sigma^r \ . \tag{30}$$

4. R/T LAWS: SPECIAL CASES

In this section, we consider special cases of Eqs.(18)–(23). Only certain simple situations in which the solutions can be obtained analytically, in a closed form, are considered.

4.1. Normal incidence

Consider an incident wave with the propagation vector $\mathbf{P}^{inc}$ perpendicular to plane interface Σ, i.e. $\mathbf{P}^{inc}$ parallel to $\mathbf{n}$, $\mathbf{P}^{inc} = |\mathbf{P}^{inc}|\mathbf{n}$. The attenuation vector $\mathbf{A}^{inc}$ may be arbitrary. Then $\mathbf{P}^{\Sigma} = \mathbf{0}$ and Eq.(22) with Eq.(18) yield:

$$\mathbf{P}^r = \mathbf{n}\,\mathrm{Re}\sigma^r \ , \quad \mathbf{A}^r = \mathbf{A}^{inc} - \mathbf{n}[(\mathbf{A}^{inc} \cdot \mathbf{n}) - \mathrm{Im}\sigma^r] \, . \tag{31}$$

Consequently, the propagation vectors $\mathbf{P}^r$ of all R/T waves are also perpendicular to Σ. The attenuation vectors, however, transform across Σ in a standard way.

Note that the interface Σ is actually parallel to the wavefront in this case, and σ^r can be determined by simpler equations of mixed specification (11), with $a_{ijkl} = a^r_{ijkl}$. They depend on the choice of the unit vector $\mathbf{m}^{inc}$ and inhomogeneity parameter D^{inc}.

4.2. Coplanar case

Consider an incident plane wave, for which the three real-valued vectors $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$ and $\mathbf{n}$ are coplanar. Planes Σ^{inc} and Σ^{att} coincide in this case, $\Sigma^{inc} = \Sigma^{att}$. It follows from the first two rules of Section 3 that vectors $\mathbf{P}^r$, $\mathbf{A}^r$ and $\mathbf{n}$ are also coplanar for any reflected/transmitted wave, and all these planes coincide. The R/T laws then constitute a planar problem.

4.3. Bulk incident plane wave ($\mathbf{A}^{inc} = \mathbf{0}$)

Assume that the incident halfspace is perfectly elastic (isotropic or anisotropic), and that the incident plane is a bulk wave. The propagation and attenuation vectors of R/T waves are then described by equations:

$$\mathbf{P}^r = \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc} \cdot \mathbf{n}) - \mathrm{Re}\sigma^r] \ , \quad \mathbf{A}^r = \mathbf{n}\,\mathrm{Im}\sigma^r \, . \tag{32}$$

The R/T waves are then either bulk waves (for $\mathrm{Im}\ \sigma^r = 0$), or inhomogeneous plane waves (for $\mathrm{Im}\ \sigma^r \neq 0$). In the latter case, however, $\mathbf{A}^r$ is always perpendicular to the interface. Homogeneous R/T plane waves cannot exist in this case.

4.4. Incident homogeneous plane wave

For homogeneous plane waves, $\mathbf{P}^{inc}$ and $\mathbf{A}^{inc}$ have the same direction, and $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$, $\mathbf{n}$ are coplanar. Consequently, $\mathbf{P}^r$, $\mathbf{A}^r$, and $\mathbf{n}$ are also coplanar, for any R/T wave. This, however, does not mean that the R/T waves are also homogeneous. Thus, a homogeneous incident plane wave generates inhomogeneous R/T plane waves, with $\mathbf{P}^r$ and $\mathbf{A}^r$ situated in the plane of incidence. Homogeneous R/T plane waves are only exceptional.

Consequently, the solution of the R/T problem in terms of only homogeneous plane waves solely is not possible, not even in the case of homogeneous incident waves. The existence of inhomogeneous R/T plane waves must be allowed.

4.5. Isotropic viscoelastic media

In this case, Eq.(7) can be solved analytically for σ^r. We obtain a simple relation

$$\sigma^r = \pm[(1/\mathcal{V}^r)^2 - \mathbf{p}^\Sigma\mathbf{p}^\Sigma]^{1/2} = \pm[(1/\mathcal{V}^r)^2 - (1/\mathcal{V}^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2]^{1/2}\,. \qquad (33)$$

Here $\mathcal{V}^r$ and $\mathcal{V}^{inc}$ are complex-valued velocities of P waves ($\mathcal{V} = \alpha$) or S waves ($\mathcal{V} = \beta$). Eq.(20) then yields:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\left[(\mathbf{p}^{inc}\cdot\mathbf{n}) \mp \sqrt{(1/\mathcal{V}^r)^2 - (1/\mathcal{V}^{inc})^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2}\right]. \qquad (34)$$

This equation is exactly the same as Eq.(2), known for isotropic perfectly elastic media. The only difference is that $\mathbf{p}^{inc}$, $\mathcal{V}^r$ and $\mathcal{V}^{inc}$ are complex-valued in this case. If we use Q^r and Q^{inc} to denote the quality factors of the R/T waves and of the incident wave, we can insert the following expressions into Eq.(34),

$$\left(\frac{1}{\mathcal{V}^r}\right)^2 = \frac{1+\mathrm{i}/Q^r}{(V^r)^2(1+(Q^r)^{-2})}\,,\quad \left(\frac{1}{\mathcal{V}^{inc}}\right)^2 = \frac{1+\mathrm{i}/Q^{inc}}{(V^{inc})^2(1+(Q^{inc})^2)}\,. \qquad (35)$$

Quantities V^r and V^{inc} on the R.H.S.'s of Eq.(35) correspond to the real-valued velocities of the R/T and incident waves, respectively.

In practical applications, the quality factors are mostly large and $Q^{-2} \ll 1$. Then we can express Eq.(35) approximately in a simpler form,

$$(1/\mathcal{V}^r)^2 \approx (1+\mathrm{i}/Q^r)/(V^r)^2\,,\quad (1/\mathcal{V}^{inc})^2 \approx (1+\mathrm{i}/Q^{inc})/(V^{inc})^2\,. \qquad 36)$$

Equation (34) can also be simply expressed in terms of propagation vectors $\mathbf{P}^{inc}$, $\mathbf{P}^r$, and attenuation vectors $\mathbf{A}^{inc}$, $\mathbf{A}^r$. It is also valid for inhomogeneous waves and for non-coplanar $\mathbf{P}^{inc}$, $\mathbf{A}^{inc}$ and $\mathbf{n}$.

In the seismological literature, considerable attention has been devoted to the R/T laws in viscoelastic isotropic media, assuming the coplanar case. See *Borcherdt (1977, 1982)*, *Aki and Richards (1980), Krebes (1983), Wennerberg (1985), Caviglia and Morro (1992),* where many other references can be found.

4.6. Isotropic media, perfectly elastic incident halfspace

Here we again consider an interface between two isotropic media; the incident halfspace is perfectly elastic and the R/T halfspace viscoelastic. The incident wave is a bulk wave ($\mathbf{A}^{inc} = \mathbf{0}$). This is the simplest case of the R/T problem for viscoelastic media studied here. All equations can be expressed in analytical form.

The slowness vector of the incident bulk wave is given by the relation

$$\mathbf{p}^{inc} = \mathbf{P}^{inc} = \mathbf{N}^{inc}/V^{inc}\,, \quad \mathbf{p}^{inc}\cdot\mathbf{n} = \cos\theta^{inc}/V^{inc}\,, \tag{37}$$

where $\theta^{inc} = i^{inc}$ is the angle of incidence given by Eq.(26). Propagation vector $\mathbf{P}^r$ and attenuation vector $\mathbf{A}^r$ of the R/T waves are then given by Eq.(32), where σ^r is given by Eq.(33). We can express Eq.(33) in the following form

$$\sigma^r = \pm[a + \mathrm{i}b]^{1/2}\,, \tag{38}$$

where

$$a = \frac{1}{(V^r)^2(1+(Q^r)^{-2})} - \frac{\sin^2\theta^{inc}}{(V^{inc})^2}, \quad b = \frac{1}{Q^r(V^r)^2(1+(Q^r)^{-2})}\,. \tag{39}$$

Quantities a and b are always real-valued, a is arbitrary and b non-negative. Consequently,

$$\mathrm{Re}\sigma^r = \pm\sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2}\,, \quad \mathrm{Im}\sigma^r = \pm\sqrt{\sqrt{a^2+b^2}-a}\Big/\sqrt{2}\,. \tag{40}$$

Equations (32) and (40) yield the final expressions for $\mathbf{P}^r$ and $\mathbf{A}^r$:

$$\mathbf{P}^r = \mathbf{N}^{inc}/V^{inc} - \mathbf{n}\left[\cos\theta^{inc}/V^{inc} \mp \sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2}\right]$$

$$\mathbf{A}^r = \pm\mathbf{n}\sqrt{\sqrt{a^2+b^2}-a}\Big/\sqrt{2}\,. \tag{41}$$

We can also compute the R/T angle θ^r, defined as the acute angle between $\mathbf{P}^r$ and $\mathbf{n}$:

$$\cos\theta^r = \mathbf{P}^r\cdot\mathbf{n}/|\mathbf{P}^r| = \pm\sqrt{\sqrt{a^2+b^2}+a}\Big/\sqrt{2}|\mathbf{P}^r|. \tag{42}$$

These are the final expressions for $\mathbf{P}^r$, $\mathbf{A}^r$ and $\cos\theta^r$ of any R/T wave. Using Eqs.(41) and (8), we can simply compute $|\mathbf{P}^r|$, $|\mathbf{A}^r|$, $\mathbf{N}^r$, $\mathbf{M}^r$, $\mathcal{C}^r$, $\cos\gamma^r$, etc. We explicitly present only the important expressions for phase velocity $\mathcal{C}^r$ and attenuation angle $\cos\gamma^r$:

$$\mathcal{C}^r = 1/|\mathbf{P}^r| = 1/[\sin^2\theta^{inc}/(V^{inc})^2 + \tfrac{1}{2}(\sqrt{a^2+b^2}+a)]^{1/2}\,,$$

$$\cos\gamma^r = b/|\mathbf{P}^r||\mathbf{A}^r|\,. \tag{43}$$

The viscoelastic properties of R/T waves are fully controlled by quality factor Q^r, hidden in a and b, particularly in b.

4.7. Perfectly elastic isotropic media

The results presented in this paragraph are mostly known, although certain interesting aspects have not been often sufficiently explained. The derivation is simple, we only insert $Q^r \to \infty$ into the equations derived in Section 4.6.

For $Q^r \to \infty$, we obtain from Eq.(39):

$$a = (1/V^r)^2 - \sin^2\theta^{inc}/(V^{inc})^2 \ , \qquad b = 0 \ . \tag{44}$$

As $b = 0$, Eq.(40) for Re σ^r and Im σ^r can be expressed in the following form:

$$\mathrm{Re}\sigma^r = \pm\sqrt{|a| + a}\Big/\sqrt{2} \ , \qquad \mathrm{Im}\sigma^r = \pm\sqrt{|a| - a}\Big/\sqrt{2} \ . \tag{45}$$

It is simple to see that the behaviour of the R/T waves is quite different for positive a (subcritical region), $a = 0$ (critical case), and negative a (postcritical region):

a) **Subcritical region**, $\sin\theta^{inc} < V^{inc}/V^r$: In this case $a > 0$. We obtain $\mathbf{A}^r = \mathbf{0}$, so that the R/T wave is a bulk wave. Phase velocity $\mathcal{C}^r$ is given by the relation, see Eqs.(43) and (44),

$$\mathcal{C}^r = 1/[\sin^2\theta^{inc}/(V^{inc})^2 + (1/V^r)^2 - \sin^2\theta^{inc}/(V^{inc})^2]^{1/2} = V^r \ . \tag{46}$$

We further insert $b = 0$ into Eq.(42) and obtain $cos\theta^r$:

$$\cos\theta^r = \pm\sqrt{|a| + a}\Big/\sqrt{2}|\mathbf{P}^r| = \pm[1 - (V^r/V^{inc})^2\sin^2\theta^{inc}]^{1/2} \ . \tag{47}$$

Equations (46) and (47) fully correspond to the familiar form of Snell's law (1), valid for bulk waves.

b) **Critical incidence**, $\sin\theta^{inc} = V^{inc}/V^r$. In this case, $a = 0$, $\theta^r = 90^0$, $\mathcal{C}^r = V^r$. The propagation vector of R/T wave is oriented along interface Σ, and the phase velocity corresponds to V^r.

c) **Postcritical region**, $\sin\theta^{inc} > V^{inc}/V^r$. In this case $a < 0$ and Re $\sigma^r = 0$. We further obtain

$$\mathbf{A}^r = \pm\mathbf{n}\sqrt{|a|}, = \pm\mathbf{n}\sqrt{\sin^2\theta^{inc}/(V^{inc})^2 - 1/(V^r)^2} \ , \quad \mathcal{C}^r = V^{inc}/\sin\theta^{inc} \ ,$$
$$\theta^r = \tfrac{1}{2}\pi \ , \quad \gamma^r = \tfrac{1}{2}\pi \ . \tag{48}$$

Thus, the propagation vector $\mathbf{P}^r$ of the R/T wave is always parallel to interface Σ in the postcritical region, for any angle of incidence, $V^{inc}/V^r < \sin\theta^{inc} < 1$. Attenuation vector $\mathbf{A}^r$ is always perpendicular to the interface, so that the R/T plane wave is inhomogeneous. The phase velocity of R/T waves in the postcritical region, however, is not constant, but depends on the angle of incidence. It decreases with increasing θ^{inc}. For the critical angle of incidence, it equals V^r, and for the grazing angle of incidence

($\theta^{inc} = 90^0$) it equals V^{inc}. A common mistake is to consider the phase velocity $\mathcal{C}^r$ of the R/T wave in the postcritical region to be V^r.

4.8. Viscoelastic monoclinic anisotropic media: SH waves

Equation (7) represents an algebraic equation of the sixth degree in σ, and must be solved numerically. For SH waves, propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) anisotropic viscoelastic medium, however, the algebraic equation of the sixth degree reduces to a quadratic equation, and may be solved analytically, similarly as in viscoelastic isotropic media. See *Červený and Pšenčík (2005)*.

We choose the Cartesian coordinate system x_i so that the plane of symmetry corresponds to plane x_1x_3, and assume $p_2^{inc} = 0$. The two quantities σ^r corresponding to R/T SH waves are then given by relations, see *Červený and Pšenčík (2005, Eq.(50))*:

$$\sigma^r = -\frac{E_{22}}{\Gamma_{22}} \pm \left[\frac{1}{\Gamma_{22}} - \frac{(p_1^\Sigma n_3 - n_1 p_3^\Sigma)^2 \Delta}{\Gamma_{22}^2}\right]^{1/2} , \tag{49}$$

where

$$\Gamma_{22} = A_{66} n_1^2 + A_{44} n_3^2 + 2A_{46} n_1 n_3 ,$$
$$E_{22} = A_{66} n_1 p_1^\Sigma + A_{44} n_3 p_3^\Sigma + A_{46}(n_1 p_3^\Sigma + p_1^\Sigma n_3) , \tag{50}$$
$$\Delta = A_{44} A_{66} - A_{46}^2,$$

and where A_{44}, A_{66} and A_{46} are the Voigt complex-valued density-normalised viscoelastic moduli, corresponding to the R/T halfspace under consideration. Quantity Δ does not depend on $\mathbf{n}$ and $\mathbf{p}^\Sigma$. Using Eq.(20), we then obtain the complex-valued slowness vector of R/T SH waves:

$$\mathbf{p}^r = \mathbf{p}^{inc} - \mathbf{n}\left\{(\mathbf{p}^{inc} \cdot \mathbf{n}) + \frac{E_{22}}{\Gamma_{22}} \mp \left[\frac{1}{\Gamma_{22}} - \frac{(p_1^\Sigma n_3 - n_1 p_3^\Sigma)^2 \Delta}{\Gamma_{22}^2}\right]^{1/2}\right\} . \tag{51}$$

Equation (51) represents the only analytical solution of the R/T problem for anisotropic viscoelastic media. All other situations require at least the solutions of the algebraic equation of the fourth degree (P and SV waves in an analogous plane of symmetry).

We shall not discuss Eq.(51) for general viscoelastic media, but only consider a simpler case of perfectly elastic anisotropic media (A_{44},A_{66} and A_{46} real-valued, and incident bulk waves ($\mathbf{A}^{inc} = \mathbf{0}$). Eq.(51) then yields:

$$\mathbf{P}^r = \mathbf{P}^{inc} - \mathbf{n}\left\{(\mathbf{P}^{inc} \cdot \mathbf{n}) + \frac{E_{22}}{\Gamma_{22}} \mp \mathrm{Re}\left[\frac{1}{\Gamma_{22}} - (p_1^\Sigma n_3 - p_3^\Sigma n_1)^2 \frac{\Delta}{\Gamma_{22}^2}\right]^{1/2}\right\} ,$$

$$\mathbf{A}^r = \pm \mathbf{n}\, \mathrm{Im}\left[1/\Gamma_{22} - (p_1^\Sigma n_3 - n_1 p_3^\Sigma)^2 \Delta / \Gamma_{22}^2\right]^{1/2} . \tag{52}$$

The expression under the square root is real-valued, and may be positive, zero or negative. The positive case (corresponding to subcritical incidence) yields bulk waves. This is the standard situation, well-known from applications. For the negative case (postcritical incidence), we obtain

$$\mathbf{P}^r = \mathbf{P}^{inc} - \mathbf{n}[(\mathbf{P}^{inc}\cdot\mathbf{n}) + E_{22}/\Gamma_{22}]\,,$$
$$\mathbf{A}^r = \pm\mathbf{n}\,\mathrm{Im}[1/\Gamma_{22} - (p_1^{\Sigma}n_3 - n_1p_3^{\Sigma})^2\Delta/\Gamma_{22}^2]^{1/2}\,. \tag{53}$$

Thus, the attenuation vector is perpendicular to reference plane Σ. Propagation vector $\mathbf{P}^r$, however, is not generally parallel to the reference plane. This is easy to see if we compute $\mathbf{P}^r\cdot\mathbf{n}$:

$$\mathbf{P}^r\cdot\mathbf{n} = -E_{22}/\Gamma_{22}\,. \tag{54}$$

Thus, $\mathbf{P}^r$ is parallel to the interface in the postcritical region only for $E_{22}=0$. This is a great difference with respect to isotropic media, and is related to the fact that the energy flux vector has a different direction than the propagation vector an anisotropic media.

5. SPECIFICATION OF INCIDENT WAVE

In the preceding sections, we have specified the slowness vector of the incident wave only in a general form, $\mathbf{p}^{inc}=\mathbf{P}^{inc}+\mathrm{i}\mathbf{A}^{inc}$. Moreover, we have not tested whether $\mathbf{p}^{inc}$ satisfies the constraint relation (4). In such a case, we have not needed to know the viscoelastic moduli of the incident halfspace at all. It has been sufficient to compute the tangential components $\mathbf{p}^{\Sigma}$ in reference plane Σ from $\mathbf{p}^{inc}$, see Eq.(16).

In this section, we discuss the R/T problem in more specific terms. We denote the complex-valued density-normalised viscoelastic moduli in the incident halfspace by a_{ijkl}^{inc}, and specify the plane incident wave by two unit vectors $\mathbf{N}^{inc}$ and $\mathbf{m}^{inc}$ and by scalar D^{inc}, where:

a) $\mathbf{N}^{inc}$ is a unit real-valued propagation vector, specifying the direction of propagation, $\mathbf{N}^{inc}=\mathbf{P}^{inc}/|\mathbf{P}^{inc}|$.
b) $\mathbf{m}^{inc}$ is a real-valued unit vector, perpendicular to $\mathbf{N}^{inc}$, specifying the propagation-attenuation plane. This plane contains both the propagation and attenuation vectors $\mathbf{P}^{inc}$ and $\mathbf{A}^{inc}$.
c) D^{inc} is the inhomogeneity parameter of the incident wave, and $|D^{inc}|$ the inhomogeneity strength of the incident wave.

The specification of $\mathbf{N}^{inc}$, $\mathbf{m}^{inc}$, and D^{inc}, for known a_{ijkl}^{inc}, is quite sufficient to determine slowness vector $\mathbf{p}^{inc}$ completely:

$$\mathbf{p}^{inc} = \sigma^{inc}\mathbf{N}^{inc} + \mathrm{i}D^{inc}\mathbf{m}^{inc}\,, \tag{55}$$

see Eq.(10). Quantity σ^{inc} is a solution of the algebraic equation of the sixth degree:

$$\det[a_{ijkl}^{inc}(\sigma^{inc}N_j^{inc} + \mathrm{i}D^{inc}m_j^{inc})(\sigma^{inc}N_l^{inc} + \mathrm{i}D^{inc}m_l^{inc}) - \delta_{ik}] = 0\,. \tag{56}$$

Consequently, the slowness vector $\mathbf{p}^{inc}$, given by Eq.(55) with Eq.(56), satisfies the constraint equation (4).

There are six roots σ^{inc} of Eq.(56), corresponding to the P, S1 and S2 plane waves, propagating towards and away from reference plane Σ in the incident halfspace. We have to select the root corresponding to the incident wave we require, propagating towards Σ. The problem of the selection criteria, however, is not treated here, see Introduction.

For completeness, we also present relations for some other useful quantities related to the incident wave, assuming $\mathbf{N}^{inc}$, $\mathbf{m}^{inc}$, D^{inc} and σ^{inc} are known:

$$\begin{gathered}\mathbf{P}^{inc} = (\mathrm{Re}\,\sigma^{inc})\mathbf{N}^{inc}\ , \quad \mathbf{A}^{inc} = (\mathrm{Im}\,\sigma^{inc})\mathbf{N}^{inc} + D^{inc}\mathbf{m}^{inc}\ , \\ |\mathbf{P}^{inc}| = |\mathrm{Re}\,\sigma^{inc}|\ , \quad |\mathbf{A}^{inc}| = [(\mathrm{Im}\,\sigma^{inc})^2 + (D^{inc})^2]^{1/2}\ , \\ \mathbf{M}^{inc} = \mathbf{A}^{inc}/|\mathbf{A}^{inc}|\ , \quad \mathcal{C}^{inc} = 1/|\mathrm{Re}\,\sigma^{inc}|\ , \\ \cos\gamma^{inc} = (\mathrm{Re}\,\sigma^{inc})(\mathrm{Im}\,\sigma^{inc})/|\mathbf{P}^{inc}||\mathbf{A}^{inc}|\ . \end{gathered} \tag{57}$$

Using Eqs.(16) and (57), we can calculate the tangential component $\mathbf{p}^{\Sigma} = \mathbf{P}^{\Sigma} + \mathrm{i}\mathbf{A}^{\Sigma}$ of the slowness vector $\mathbf{p}^{inc}$ of the incident wave into interface Σ:

$$\begin{gathered}\mathbf{P}^{\Sigma} = (\mathrm{Re}\ \sigma^{inc})(\mathbf{n} \times (\mathbf{N}^{inc} \times \mathbf{n}))\ , \\ \mathbf{A}^{\Sigma} = (\mathrm{Im}\ \sigma^{inc})(\mathbf{n} \times (\mathbf{N}^{inc} \times \mathbf{n})) + D^{inc}(\mathbf{n} \times (\mathbf{m}^{inc} \times \mathbf{n}))\ . \end{gathered} \tag{58}$$

The expressions (58) can be simplified if we introduce two real-valued vectors $\mathbf{N}^{\Sigma}$ and $\mathbf{m}^{\Sigma}$, representing the tangential projections of $\mathbf{N}^{inc}$ and $\mathbf{m}^{inc}$ into Σ:

$$\mathbf{N}^{\Sigma} = \mathbf{n} \times (\mathbf{N}^{inc} \times \mathbf{n})\ , \quad \mathbf{m}^{\Sigma} = \mathbf{n} \times (\mathbf{m}^{inc} \times \mathbf{n})\ . \tag{59}$$

Note that $\mathbf{N}^{\Sigma}$ and $\mathbf{m}^{\Sigma}$ are not generally unit vectors, and are perpendicular to $\mathbf{n}$; so that $\mathbf{n} \cdot \mathbf{N}^{\Sigma} = \mathbf{n} \cdot \mathbf{m}^{\Sigma} = 0$. Eqs.(58) then read

$$\mathbf{P}^{\Sigma} = (\mathrm{Re}\sigma^{inc})\mathbf{N}^{\Sigma}\ , \quad \mathbf{A}^{\Sigma} = (\mathrm{Im}\sigma^{inc})\mathbf{N}^{\Sigma} + D^{inc}\mathbf{m}^{\Sigma}\ . \tag{60}$$

If $\mathbf{m}^{inc}$ lies in the plane of incidence, i.e. if $\mathbf{N}^{\Sigma}$, $\mathbf{m}^{\Sigma}$ and $\mathbf{n}$ are coplanar, the vectors $\mathbf{N}^{\Sigma}$ and $\mathbf{m}^{\Sigma}$ are parallel. Consequently, also $\mathbf{P}^{\Sigma}$ and $\mathbf{A}^{\Sigma}$ are parallel in this case, although the incident wave may be inhomogeneous.

Using Eq.(19), we obtain an expression for $\mathbf{p}^{r}$ in terms of $\mathbf{p}^{\Sigma}$ or $\mathbf{p}^{inc}$, corresponding to any R/T wave:

$$\mathbf{p}^{r} = \sigma^{r}\mathbf{n} + \mathbf{p}^{\Sigma} = \sigma^{r}\mathbf{n} + \sigma^{inc}\mathbf{N}^{\Sigma} + \mathrm{i}D^{inc}\mathbf{m}^{\Sigma}\ , \tag{61}$$

where σ^{r} is a solution of Eq.(21) and σ^{inc} a relevant solution of Eq.(56). Analogously, we obtain expressions for $\mathbf{P}^{r}$ and $\mathbf{A}^{r}$

$$\mathbf{P}^{r} = (\mathrm{Re}\,\sigma^{r})\mathbf{n} + \mathbf{P}^{\Sigma}\ , \quad \mathbf{A}^{r} = (\mathrm{Im}\,\sigma^{r})\mathbf{n} + \mathbf{A}^{\Sigma}\ , \tag{62}$$

where $\mathbf{P}^{\Sigma}$, $\mathbf{A}^{\Sigma}$ are given by Eq.(60). Using Eq.(60), we can also express $\mathbf{P}^{r}$ and $\mathbf{A}^{r}$ directly in terms related directly to the incident wave (55):

$$\mathbf{P}^r = (\operatorname{Re}\sigma^r)\mathbf{n} + (\operatorname{Re}\sigma^{inc})\mathbf{N}^\Sigma \ , \quad \mathbf{A}^r = (\operatorname{Im}\sigma^r)\mathbf{n} + (\operatorname{Im}\sigma^{inc})\mathbf{N}^\Sigma + D^{inc}\mathbf{m}^\Sigma \, . \tag{63}$$

Equation (61) (or alternatively Eq.(63)), with Eq.(59), represent the final expressions for $\mathbf{p}^r$, $\mathbf{P}^r$ and $\mathbf{A}^r$. Now we shall discuss the properties of R/T waves in a greater detail. We are mainly interested in the phase velocity $\mathcal{C}^r$, in the unit vector $\mathbf{N}^r$ perpendicular to the wavefront, in the inhomogeneity parameter D^r and corresponding unit vector $\mathbf{m}^r$, of the R/T wave. In the derivations, we shall consider the relations $\mathbf{N}^\Sigma \cdot \mathbf{N}^\Sigma = (\mathbf{N}^{inc} \cdot \mathbf{n})^2$, $\mathbf{m}^\Sigma \cdot \mathbf{m}^\Sigma = (\mathbf{m}^{inc} \cdot \mathbf{n})^2$, following from Eq.(59).

From Eq.(63), we easily obtain

$$|\mathbf{P}^r| = [(\operatorname{Re}\sigma^r)^2 + (\operatorname{Re}\sigma^{inc})^2(\mathbf{N}^{inc}\cdot\mathbf{n})^2]^{1/2} \, ,$$

$$|\mathbf{A}^r| = [(\operatorname{Im}\sigma^r)^2 + (\operatorname{Im}\sigma^{inc})^2(\mathbf{N}^{inc}\cdot\mathbf{n})^2 + (D^{inc})^2(\mathbf{m}^{inc}\cdot\mathbf{n})^2$$
$$-2(\operatorname{Im}\sigma^{inc})D^{inc}(\mathbf{N}^{inc}\cdot\mathbf{n})(\mathbf{m}^{inc}\cdot\mathbf{n})]^{1/2} \, . \tag{64}$$

As $|\mathbf{P}^r| = 1/\mathcal{C}^r$, the phase velocity $\mathcal{C}^r$ of the R/T wave is given by the relation:

$$\mathcal{C}^r = 1/|\mathbf{P}^r| = [(\operatorname{Re}\sigma^r)^2 + (\operatorname{Re}\sigma^{inc})^2(\mathbf{N}^{inc}\cdot\mathbf{n})^2]^{-1/2} \, . \tag{65}$$

The unit vector $\mathbf{N}^r$ perpendicular to the wavefront of the R/T wave is expressed as follows:

$$\mathbf{N}^r = \mathbf{P}^r/|\mathbf{P}^r| = \mathcal{C}^r[(\operatorname{Re}\sigma^r)\mathbf{n} + (\operatorname{Re}\sigma^{inc})\mathbf{N}^\Sigma] \, . \tag{66}$$

The projection of the attenuation vector $\mathbf{A}^r$ of the R/T wave into the wavefront of that wave is given by the relation, see Eq.(13),

$$D^r\mathbf{m}^r = \mathbf{N}^r \times (\mathbf{A}^r \times \mathbf{N}^r) \, . \tag{67}$$

For $\mathbf{A}^r$ parallel with $\mathbf{N}^r$, the inhomogeneity parameter $D^r = 0$, and the R/T wave is homogeneous. In this case, the unit vector $\mathbf{m}^r$ is not defined. For $\mathbf{A}^r$ non-parallel to $\mathbf{N}^r$, the inhomogeneity parameter D^r is given by the relation:

$$D^r = |\mathbf{N}^r \times (\mathbf{A}^r \times \mathbf{N}^r)| \, , \tag{68}$$

and the unit vector $\mathbf{m}^r$ by the relation:

$$\mathbf{m}^r = (\mathbf{N}^r \times (\mathbf{A}^r \times \mathbf{N}^r))/D^r \, . \tag{69}$$

Alternatively, the signs of both D^r and $\mathbf{m}^r$ may be taken opposite.

For isotropic viscoelastic media, the final form of R/T laws may be fully expressed analytically. We merely use

$$\sigma^{inc} = \left[\left(1/\mathcal{V}^{inc}\right)^2 + \left(D^{inc}\right)^2\right]^{1/2} , \tag{70}$$

so that

$$\mathbf{p}^{inc} = \left[\left(1/\mathcal{V}^{inc}\right)^2 + \left(D^{inc}\right)^2\right]^{1/2} \mathbf{N}^{inc} + \mathrm{i}D^{inc}\mathbf{m}^{inc} \, . \tag{71}$$

We further use

$$\sigma^r = \pm\left[(1/\mathcal{V}^r)^2 - \left(1/\mathcal{V}^{inc}\right)^2 + (\mathbf{p}^{inc}\cdot\mathbf{n})^2\right]^{1/2} . \tag{72}$$

Then

$$\mathbf{p}^r = \mathbf{P}^r + \mathrm{i}\mathbf{A}^r , \tag{73}$$

where $\mathbf{p}^r$, $\mathbf{P}^r$ and $\mathbf{A}^r$ are given by Eqs.(61) and (63), with Eq.(59). All other expressions (64)–(69) for the quantities related to R/T waves remain the same as in general case. Explicit form of these expressions may be written in many alternative forms.

The expressions simplify considerably for $D^{inc} = 0$ (homogeneous incident wave). In this case, we obtain

$$\sigma^{inc} = \tfrac{1}{\mathcal{V}^{inc}} , \quad \sigma^r = \pm\left[(1/\mathcal{V}^r)^2 - \left(1/\mathcal{V}^{inc}\right)^2 \sin^2\theta^{inc}\right]^{1/2} . \tag{74}$$

It is simple to conclude from Eq.(63) (with $D^{inc} = 0$) and Eq.(68) that the R/T waves are in general inhomogeneous also if the incident wave is homogeneous.

For SH waves in viscoelastic monoclinic anisotropic media, the final form of R/T laws may be again expressed analytically, see Section 4.8. The slowness vector of the incident wave is expressed using Eq.(55), where σ^{inc} is given by the relation:

$$\sigma^{inc} = -\mathrm{i}D^{inc}\Lambda/\Gamma_{22} \pm [1/\Gamma_{22} + (D^{inc})^2\Delta/\Gamma_{22}^2]^{1/2} . \tag{75}$$

Here Δ is given by Eq.(50), Λ and Γ_{22} by the relations:

$$\Gamma_{22} = A_{66}(N_1^{inc})^2 + A_{44}(N_3^{inc})^2 + 2A_{46}N_1^{inc}N_3^{inc} ,$$
$$\Lambda = (A_{66} - A_{44})N_1^{inc}N_3^{inc} + A_{46}((N_3^{inc})^2 - (N_1^{inc})^2) . \tag{76}$$

For the derivation of Eq.(75), see *Červený and Pšenčík (2005, Eq. (57))*.

Final expressions for $\mathbf{p}^r$, $\mathbf{P}^r$ and $\mathbf{A}^r$ are then given by Eqs.(61) and (63), where σ^r is given by Eq.(49) and σ^{inc} by Eq.(75). All other expressions for $|\mathbf{P}^r|$, $|\mathbf{A}^r|$, $\mathcal{C}^r$, D^r, $\mathbf{N}^r$ and $\mathbf{m}^r$ remain the same as in Eqs.(64)–(69).

By a proper choice of elastic moduli A_{44}, A_{66} and A_{46}, the relations can be used for SH waves propagating in orthorhombic and hexagonal anisotropic viscoelastic media.

6. CONCLUDING REMARKS

The reflection/transmission laws at a plane interface between two viscoelastic anisotropic halfspaces are derived and discussed. Under the R/T laws, we understand the procedures for determining the slowness vectors of R/T plane waves from the slowness vector of the incident wave. The slowness vectors of all waves are, in general, complex-valued. The slowness vector of inhomogeneous incident wave is expressed in terms of the so-called mixed specification, using the unit vectors $\mathbf{N}^{inc}$, $\mathbf{m}^{inc}$ and the inhomogeneity parameter D^{inc}. The derived R/T laws are quite general; they are valid for isotropic and anisotropic, viscoelastic and perfectly elastic media, for bulk, homogeneous and inhomogeneous incident plane waves, and for any type of reflected and transmitted plane waves. All generated R/T waves are, in general, inhomogeneous, even if the incident wave is homogeneous.

The derivation of R/T laws given in Section 5 requires, in general, the solution of two algebraic equations of the sixth degree; one corresponding to the mixed specification of the slowness vector of the incident wave, and the other corresponding to the componental specification of the slowness vector of the R/T waves under consideration. In special cases, the algebraic equations of the sixth degree reduce to the algebraic equations of the fourth degree, or even of the second degree. For isotropic viscoelastic media, and for SH waves in the plane of symmetry of monoclinic (orthorhombic, hexagonal) anisotropic viscoelastic media, the algebraic equation of the sixth degree reduce to the second degree and can be fully solved analytically.

The derivation of the R/T laws is based on the fact that the projections of the complex-valued slowness vectors of the incident and all relevant R/T waves on the interface must be the same. This requirement should be valid both for waves propagating towards and away from the interface.

The same algorithms of determining the R/T laws can be used also for poroviscoelastic media. The only difference is that the algebraic equations for σ^{inc} and σ^{r} are of the eight degree (not sixth).

The presented algorithm of the R/T laws is a necessary prerequisite for the computation of R/T coefficient at a plane interface between two viscoelastic anisotropic media. In such case, a proper selection algorithm must be also supplemented, which would be able to distinguish the waves propagating towards and away from the interface. In viscoelastic anisotropic media, such selection algorithms are not straightforward, and are not treated here. For a detailed discussion of the selection criteria see *Ruud (2006)* and *Krebes and Daley (2007)*, where many other references can be found.

Acknowledgements: This research has been supported by the Consortium Project “Seismic Waves in Complex 3-D Structures”, and by Research Projects 205/04/1104 and 205/05/2182 of the Grant Agency of the Czech Republic.

APPENDIX
SNELL’S LAW IN VISCOELASTIC ANISOTROPIC MEDIA

We have presented two forms of Snell’s law:

1. The **first form** of Snell’s law is represented by its classical version (1),

$$\sin i^{inc}/V^{inc} = \sin i^{r}/V^{r} \; . \tag{A.1}$$

   Eq.(A.1) is valid for isotropic perfectly elastic media. V^{inc} and V^{r} are real-valued velocities (α or β) in the incident and R/T halfspace, respectively, i^{inc} is the angle of incidence (i.e. the acute angle between the normal to the interface and the slowness vector of the incident wave), and i^{r} is the reflection/transmission angle, defined in an analogous way. Angle i^{r}, however, may be complex-valued for $\sin i^{inc} > V^{inc}/V^{r}$ (postcritical incidence).

2. The **second form** of the Snell's law, see Eq.(27),

$$\sin\theta^{inc}/\mathcal{C}^{inc} = \sin\theta^{r}/\mathcal{C}^{r} \; , \tag{A.2}$$

is valid quite generally, for viscoelastic and perfectly elastic media, for anisotropic and isotropic media, for inhomogeneous, and homogeneous plane waves. In certain cases, the individual quantities in Eq.(A.2) have a different meaning than in Eq.(A.1). θ^{inc} and θ^{r} are always real-valued and determine the directions of the propagation vectors $\mathbf{P}^{inc}$ and $\mathbf{P}^{r}$ of the incident and R/T waves, see Eq.(26), $\mathcal{C}^{inc}$ and $\mathcal{C}^{r}$ are the relevant real-valued phase velocities of both waves. Phase velocities $\mathcal{C}^{inc}$ and $\mathcal{C}^{r}$ are not constants, but they depend on the directions of $\mathbf{P}^{inc}$ and $\mathbf{A}^{inc}$, and on inhomogeneity parameter D^{inc}.

The second form of Snell's law (A.2) is considerably more general than the first form. Unfortunately, it is not constructive as Eq.(A.1). In Eq.(A.1), three quantities are often known, and the fourth may be computed. For example, V^{inc}, i^{inc} and i^{r} are known, and V^{r} is computed. Alternatively, V^{inc}, V^{r} and i^{inc} are known, and i^{r} is computed. This is, in general, impossible in Eq.(A.2). Even if we specify θ^{inc} and compute the corresponding $\mathcal{C}^{inc}$, both θ^{r} and $\mathcal{C}^{r}$ remain unknown. To compute them, the procedures proposed in Sections 3 and 4 must be used. Snell's law (A.2) is not sufficient for this purpose. In fact, Snell's law (A.2) is not needed in this procedure at all, it is only obtained as a by-product.

In isotropic perfectly elastic media, however, both Eq.(A.1) and Eq.(A.2) yield the same results. For subcritically incident waves the equivalence of Eq.(A.1) and Eq.(A.2) is obvious, as $\mathcal{C}^{inc} = V^{inc}$, $\mathcal{C}^{r} = V^{r}$, $\sin\theta^{inc} = \sin i^{inc}$ and $\sin\theta^{r} = \sin i^{r}$ in this case, see Section 4.7, Paragraph a. For postcritically incident waves, however, this is not immediately obvious. The reader is reminded that i^{r} is complex-valued in this case, but θ^{r} is real-valued.

We can, however, simply prove it even in this case. For incident wave, we again have $\mathcal{C}^{inc} = V^{inc}$ and $\theta^{inc} = i^{inc}$. For reflected waves, we use Eqs.(48): $\mathcal{C}^{r} = V^{inc}/\sin\, i^{inc}$, $\theta^{r} = \frac{1}{2}\pi$,. This yields

$$\sin\,\theta^{r}/\mathcal{C}^{r} = \sin\, i^{inc}/V^{inc} \; . \tag{A.3}$$

Consequently, the second form of the Snell's law (A.2) is identically satisfied. The complex-valued R/T angle i^{r} is given by the simple relation:

$$\sin\, i^{r} = V^{r}\sin\, i^{inc}/V^{inc} \; . \tag{A.4}$$

For viscoelastic anisotropic media, however, the classical version (A.1) of Snell's law is not valid.

References

Aki K. and Richards P., 1980. *Quantitative Seismology. Theory and Methods*. Freeman, San Francisco.

Borcherdt R.D., 1977. Reflection and refraction of type-II S waves in elastic and anelastic media. *Bull. Seismol. Soc. Amer.*, **67**, 43–67.

Borcherdt R.D., 1982. Reflection-refraction of general P- and type-I S waves in elastic and anelastic solids. *Geophys. J. R. Astr. Soc.*, **70**, 621–638.

Carcione J.M., 2001. *Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic and Porous Media*. Pergamon, Amsterdam.

Caviglia G. and Morro A., 1992. *Inhomogeneous Waves in Solids And Fluids*. World Scientific, Singapore.

Caviglia G. and Morro A., 1999. Existence and uniqueness in the reflection-transmission problem. *Q. J. Mech. Appl. Math.*, **52**, 543–564.

Červený V., 2001. *Seismic Ray Theory*. Cambridge Univ. Press, Cambridge.

Červený V., Molotkov I.A. and Pšenčík I., 1977. *Ray Method in Seismology*. Charles University, Praha.

Červený V. and Pšenčík I., 2005. Plane waves in viscoelastic anisotropic media - I. Theory. *Geophys. J. Int.*, **161**, 197–212.

Declercq N.F., Briers R., Degrieck J. and Leroy O., 2005. The history and properties of ultrasonic inhomogeneous waves. *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, **52**, 775–791.

Deschamps M., 1994. Reflection and refraction of the evanescent plane wave on plane interfaces. *J. Acoust. Soc. Am.*, **96**, 2841–2848.

Deschamps M., Poirée B. and Poncelet O., 1997. Energy velocity of complex harmonic plane waves in viscous fluids. *Wave Motion*, **25**, 51–60.

Fedorov F.I., 1968. *Theory of Elastic Waves in Crystals*. Plenum, New York.

Gajewski D. and Pšenčík I., 1987. Computation of high-frequency seismic wavefields in 3-D laterally inhomogeneous anisotropic media. *Geophys. J. R. Astr. Soc.*, **91**, 383–411.

Henneke E.G., 1972. Reflection-refraction of a stress wave at a plane boundary between anisotropic media. *J. Acoust. Soc. Am.*, **51**, 210–217.

Krebes E.S., 1983. The viscoelastic reflection/transmission problem, two special cases. *Bull. Seismol. Soc. Amer.*, **73**, 1673–1683.

Krebes E.S. and Daley P.F., 2007. Difficulties with computing anelastic plane wave reflection and transmission coefficients. *Geophys. J. Int.*, doi: 10.1111/j.1365-246X.2006.03349.x.

Robein E., 2003. *Velocities, Time-Imaging and Depth-Imaging in Reflection Seismics. Principles and Methods*. EAGE Publications, Houten, The Netherlands.

Ruud B.O., 2006. Ambiguous reflection coefficients for anelastic media. *Stud. Geophys. Geod.*, **50**, 479–498.

Wennerberg L., 1985. Snell’s law for viscoelastic materials. *Geophys. J. R. Astr. Soc.*, **81**, 13–18.