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Plane waves in viscoelastic anisotropic media—I. Theory

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SUMMARY

Properties of homogeneous and inhomogeneous plane waves propagating in an unbounded viscoelastic anisotropic medium in an arbitrarily specified direction **N** are studied analytically. The method used for their calculation is based on the so-called mixed specification of the slowness vector. It is quite universal and can be applied to homogeneous and inhomogeneous plane waves propagating in perfectly elastic or viscoelastic, isotropic or anisotropic media. The method leads to the solution of a complex-valued algebraic equation of the sixth degree. Standard methods can be used to solve the algebraic equation. Once the solution has been found, the phase velocities, exponential decays of amplitudes, attenuation angles, polarization vectors, etc., of *P*, *S1* and *S2* plane waves, propagating along and against **N**, can be easily determined.

Although the method can be used for an unrestricted anisotropy, a special case of *P*, *SV* and *SH* plane waves, propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium is discussed in greater detail. In this plane the waves can be studied as functions of propagation direction **N** and of the real-valued inhomogeneity parameter *D*. For inhomogeneous plane waves, $D \neq 0$, and for homogeneous plane waves, $D = 0$. The use of the inhomogeneity parameter *D* offers many advantages in comparison with the conventionally used attenuation angle γ. In the **N**, *D* domain, any combination of **N** and *D* is physically acceptable. This is, however, not the case in the **N**, γ domain, where certain combinations of **N** and γ yield non-physical solutions. Another advantage of the use of inhomogeneity parameter *D* is the simplicity and universality of the algorithms in the **N**, *D* domain.

Combined effects of attenuation and anisotropy, not known in viscoelastic isotropic media or purely elastic anisotropic media, are studied. It is shown that, in anisotropic viscoelastic media, the slowness vector and the related quantities are not symmetrical with respect to $D = 0$ as in isotropic viscoelastic media. The phase velocity of an inhomogeneous plane wave may be higher than the phase velocity of the relevant homogeneous plane wave, propagating in the same direction **N**. Similarly, the modulus of the attenuation vector of an inhomogeneous plane wave may be lower than that for the relevant homogeneous plane wave. The amplitudes of inhomogeneous plane waves in anisotropic viscoelastic media may increase exponentially in the direction of propagation **N** for certain *D*. The attenuation angle γ cannot exceed its boundary value, γ^*. The boundary attenuation angle γ^* is, in general, different from 90°, and depends both on the direction of propagation **N** and on the sign of the inhomogeneity parameter *D*. The polarization of *P* and *SV* plane waves is, in general, elliptical, both for homogeneous and inhomogeneous waves. Simple quantitative expressions or estimates for all these effects (and for many others) are presented. The results of the numerical treatment are presented in a companion paper (Paper II, this issue).

Key words: attenuation, seismic anisotropy, seismic waves, viscoelasticity.

1 INTRODUCTION

In this paper, we study the properties of homogeneous and inhomogeneous plane waves propagating in an unbounded viscoelastic anisotropic medium in an arbitrarily specified direction. The complex-valued slowness vectors of such plane waves and their polarization vectors may be determined in several ways. Individual approaches to the solution of this problem differ in the way in which the slowness vector of the plane wave under consideration is specified. We consider a time-harmonic plane wave

$$u_j(x_k, t) = U_j \exp[-i\omega(t - p_n x_n)], \tag{1}$$

where u_j, p_j and U_j are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, respectively. Moreover, t is time and ω is a fixed, positive circular frequency. Eq. (1) represents a plane wave if, and only if, U_j and p_j are chosen in such a way that (1) satisfies the elastodynamic equation. This requirement yields the relations

$$\Gamma_{ik}(p_n)U_k = U_i, \quad i = 1, 2, 3. \tag{2}$$

The condition of solvability of the system of linear eqs (2) for U_1, U_2, U_3 reads

$$\det[\Gamma_{ik}(p_n) - \delta_{ik}] = 0. \tag{3}$$

The 3 × 3 complex-valued matrix $\Gamma_{ik}(p_n)$ is given by the relation

$$\Gamma_{ik}(p_n) = a_{ijkl} p_j p_l, \tag{4}$$

where a_{ijkl} are complex-valued, frequency-dependent, density-normalized viscoelastic moduli. The matrix $\Gamma_{ik}(p_n)$ is here referred to as the *generalized Christoffel matrix*, in contrast to the well-known *Christoffel matrix*, given by the relation

$$\Gamma_{ik}(N_n) = a_{ijkl} N_j N_l, \tag{5}$$

where N_i are the Cartesian components of real-valued unit vector $\mathbf{N}$ (Musgrave 1970; Helbig 1994). Note that the generalized Christoffel matrix has been broadly used in the seismic ray method, where it is simply called the Christoffel matrix (see Červený 2001). Here, however, we shall strictly distinguish between (4) and (5).

Eqs (2) and (3) are the *constraint relations* imposed on slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, which must be satisfied by any time-harmonic plane wave. Eq. (3) plays a basic role in determining slowness vector $\mathbf{p}$, and eq. (2) in determining polarization vector $\mathbf{U}$.

As an alternative to a_{ijkl}, we also use the complex-valued density-normalized viscoelastic moduli in the Voigt notation, $A_{\alpha\beta}(\alpha, \beta = 1, 2, \ldots, 6)$. Throughout the paper, we assume that the 6 × 6 matrix Re$A_{\alpha\beta}$ is positive definite and the 6 × 6 matrix Im$A_{\alpha\beta}$ is negative definite or zero.

We now introduce certain notations used for slowness vector $\mathbf{p}$. It is complex-valued:

$$\mathbf{p} = \mathbf{P} + i\mathbf{A}. \tag{6}$$

Here $\mathbf{P}$ is the real-valued propagation vector (perpendicular to the plane of constant phase) oriented in the direction of the propagation of the wave front (briefly the direction of propagation), $\mathbf{A}$ is the real-valued attenuation vector (perpendicular to the plane of constant amplitude), oriented in the direction of the maximum decay of amplitude. We also introduce the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ in the directions of $\mathbf{P}$ and $\mathbf{A}$, phase velocity $\mathcal{C}$, attenuation–propagation ratio δ and attenuation (or inhomogeneity) angle γ as

$$\mathbf{N} = \mathbf{P}/|\mathbf{P}|, \quad \mathbf{M} = \mathbf{A}/|\mathbf{A}|, \tag{7}$$

$$\mathcal{C} = 1/|\mathbf{P}|, \quad \delta = |\mathbf{A}|/|\mathbf{P}|, \quad \cos\gamma = \mathbf{N}\cdot\mathbf{M}. \tag{8}$$

The plane waves are called homogeneous for $\gamma = 0$ and inhomogeneous for $\gamma \neq 0$. For inhomogeneous plane waves, the plane specified by unit vectors $\mathbf{N}$ and $\mathbf{M}$ is called the *propagation–attenuation plane*, and is denoted by $\Sigma^{\parallel}$.

Note that a notation analogous to (6) has often been used in the seismological literature for wave vector $\mathbf{k} = \omega\mathbf{p}$, $\mathbf{k} = \mathbf{P} + i\mathbf{A}$ (see, for example Auld (1973), Aki & Richards (1980, p. 183), Borcherdt (1973, 1977), Krebes (1983), Krebes & Le (1994), Carcione & Cavallini (1995) and Carcione (2001)). The constraint relation (3) for wave vector $\mathbf{k}$ reads $\det[a_{ijkl}k_j k_l - \omega^2\delta_{ik}] = 0$, and is usually referred to as the *dispersion relation*. As $\mathbf{k} = \omega\mathbf{p}$, the differences between $\mathbf{k}$ and $\mathbf{p}$ are only formal, if we consider time-harmonic plane waves with $\omega > 0$ and fixed. In this paper, we use slowness vector $\mathbf{p}$ and eqs (3) and (6) systematically, and avoid wave vector $\mathbf{k}$ completely.

Now we introduce the three specifications of slowness vector $\mathbf{p}$, discussed in this paper.

In the *directional specification* of slowness vector $\mathbf{p}$, we express the slowness vector in terms of known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$,

$$\mathbf{p} = \mathcal{C}^{-1}(\mathbf{N} + i\delta\mathbf{M}). \tag{9}$$

The unknown quantities $\mathcal{C}$ and δ must be determined by inserting (9) into (3). Note that attenuation angle γ is assumed to be known, as $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$ (see eq. 8). For more details see Section 2.1.

The directional specification (9) has been used extensively in the seismological literature to study inhomogeneous plane waves propagating in viscoelastic isotropic media (see, for example, Buchen (1971), Borcherdt (1973, 1977), Aki & Richards (1980), Krebes (1983), Borcherdt & Wennerberg (1985), Borcherdt *et al.* (1986), Caviglia & Morro (1992), Brokešová & Červený (1998) and Carcione (2001)). For inhomogeneous plane waves in viscoelastic anisotropic media see Romeo (1994), Krebes & Le (1994), Carcione & Cavallini (1995), Carcione (2001) and Červený (2004).

In the *componental specification*, slowness vector $\mathbf{p}$ is expressed in terms of a known real-valued unit vector $\mathbf{n}$ and a complex-valued vector $\mathbf{p}^{\Sigma}$ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma}, \quad \text{with} \quad \mathbf{p}^{\Sigma}\cdot\mathbf{n} = 0. \tag{10}$$

In fact, $\mathbf{p}^{\Sigma}$ represents a known vectorial component of slowness vector $\mathbf{p}$ in the plane Σ perpendicular to $\mathbf{n}$. The unknown complex-valued quantity σ must be determined by inserting (10) into (3). For more details see Section 2.2.

The componental specification (10) has been traditionally used in seismology for perfectly elastic media to solve reflection/transmission problems and to compute the displacement–stress (or velocity–stress) propagator matrices (see, for example, Fedorov (1968), Gajewski & Pšenčík (1987) and Červený (2001, Section 5.4.7) for the solution of the reflection/transmission problem on an interface between two perfectly elastic anisotropic media and Woodhouse (1974), Kennett (1983, 2001), Frazer & Fryer (1989), Chapman (1994, 2004) and Thomson (1996a,b) for the computations of the displacement–stress propagator matrices). In the mentioned books and papers many other references can be also found. In certain papers using the componental specification, complex-valued moduli a_{ijkl} are considered (Frazer & Fryer 1989). A fully analogous approach has been proposed independently by Stroh (1962), and is usually referred to as the Stroh formalism. The most detailed description of the Stroh formalism can be found in Ting (1996). The Stroh formalism, popular mainly in applied mathematics and mechanics, has been applied to the study of homogeneous and inhomogeneous plane waves propagating in perfectly elastic, viscoelastic and thermoviscoelastic anisotropic media (see Shuvalov & Scott 1999, 2000; Shuvalov 2001), and to the reflection/transmission problem of viscoelastic anisotropic media (see Caviglia & Morro 1999). Let us mention that Caviglia & Morro (1999) also study the time-averaged energy flux of inhomogeneous plane waves in viscoelastic anisotropic media.

The *mixed specification* of slowness vector $\mathbf{p}$ is a special case of the componental specification (10), with $\mathbf{p}^{\Sigma}$ purely imaginary,

$$\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}. \tag{11}$$

Here $\mathbf{m}$ is a real-valued unit vector, perpendicular to $\mathbf{n}$, and D is a scalar, real-valued quantity, here referred to as the *inhomogeneity parameter*. Its absolute value, $|D|$, measures the inhomogeneity strength of the plane wave. For $D = 0$, the plane wave is homogeneous, and for $D \neq 0$ it is inhomogeneous. The slowness vector $\mathbf{p}$ is then expressed as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m}, \quad \text{with} \quad \mathbf{m}\cdot\mathbf{n} = 0. \tag{12}$$

The unknown complex-valued quantity σ must be determined by inserting (12) into (3). For more details see Section 2.3.

In the mixed specification, the plane Σ represents the plane of constant phase (wave front), as $\mathrm{Re}(\mathbf{p}^{\Sigma}) = \mathbf{0}$ along it. Consequently, $\mathbf{N} = \pm\mathbf{n}$. It is easy to see that $D\mathbf{m}$ represents the vectorial component of attenuation vector $\mathbf{A}$ in the wave front, $D\mathbf{m} = \mathbf{n}\times(\mathbf{A}\times\mathbf{n})$. This shows that unit vectors $\mathbf{n}$ and $\mathbf{m}$, which are assumed to be known, define uniquely the propagation–attenuation plane $\Sigma^{\parallel}$, in which the propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ are also situated. The mixed specification (12) was proposed in Červený (2004) for the study of homogeneous and inhomogeneous plane waves propagating in viscoelastic anisotropic media.

In this paper we derive and discuss the expressions for complex-valued slowness vector $\mathbf{p}$ of plane waves, using all the above three specifications, (9), (10) and (12), of the slowness vector. In Section 2, we consider plane waves propagating in a general viscoelastic anisotropic medium (see also Červený 2004). In Section 3, we specify these equations for the *SH*, *P* and *SV* inhomogeneous and homogeneous plane waves propagating in the symmetry plane of a monoclinic viscoelastic medium, or in any anisotropic medium with higher symmetry than monoclinic. In both sections, the results obtained under different specifications of the slowness vector are mutually compared and discussed. The derived equations also offer certain interesting physical conclusions related to the properties of *SH*, *P* and *SV* inhomogeneous plane waves (see Sections 3.4 and 3.5). For a numerical treatment see Červený & Pšenčík (2005).

Throughout this paper, we consider plane waves propagating in homogeneous media. The conclusions of the paper concerning the behaviour of slowness vectors, polarization vectors and other characteristics of wave propagation, however, also hold for local characteristics of waves propagating in general inhomogeneous viscoelastic anisotropic media.

2 VISCOELASTIC ANISOTROPIC MEDIA

In this section, we apply the three specifications of slowness vector $\mathbf{p}$ to the determination of the complex-valued slowness vector $\mathbf{p}$ of a plane wave propagating in an arbitrary unbounded anisotropic viscoelastic medium.

2.1 Directional specification of the slowness vector

Inserting the directional specification of the slowness vector (9) into constraint relations (2) and (3) yields a system of linear equations for the polarization vector $\mathbf{U}$:

$$a_{ijkl}(N_j + \mathrm{i}\delta M_j)(N_l + \mathrm{i}\delta M_l)U_k = \mathcal{C}^2 U_i, \quad i = 1, 2, 3, \tag{13}$$

and the condition of solvability of the system (13):

$$\det\left[a_{ijkl}(N_j + \mathrm{i}\delta M_j)(N_l + \mathrm{i}\delta M_l) - \mathcal{C}^2\delta_{ik}\right] = 0. \tag{14}$$

For known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$, eq. (14) represents two real-valued coupled equations for δ and $\mathcal{C}^2$. These equations are coupled polynomials of the third degree in $\mathcal{C}^2$ and of the sixth degree in δ (see Romeo 1994). In the general case of $\mathbf{N} \neq \mathbf{M}$ and of complex-valued

a_{ijkl}, the solution of (14) for δ and $\mathcal{C}^2$ is not simple. It simplifies only in exceptional cases. For example, for homogeneous waves ($\mathbf{N} = \mathbf{M}$), eq. (14) yields

$$\det\left[a_{ijkl}N_jN_l - [\mathcal{C}/(1+\mathrm{i}\delta)]^2\delta_{ik}\right] = 0. \tag{15}$$

Thus, for homogeneous waves, $\mathcal{C}^2/(1+\mathrm{i}\delta)^2$ are the eigenvalues of the 3×3 complex-valued Christoffel matrix $a_{ijkl}N_jN_l$, which can be determined by conventional methods. Quantities $\mathcal{C}^2$ and δ can then be computed from these complex-valued eigenvalues. The next two cases, which can be treated in a simpler way, correspond to inhomogeneous *P* and *S* plane waves propagating in isotropic viscoelastic media, and to inhomogeneous *SH* plane waves propagating in a symmetry plane of an monoclinic (orthorhombic, hexagonal) viscoelastic medium (see Section 3.1).

In addition to the complicated solution of (14) for $\mathcal{C}^2$ and δ, eq. (14) has another great disadvantage. For certain combinations of $\mathbf{N}$ and $\mathbf{M}$, it may yield non-physical solutions, $\mathcal{C}^2 < 0$.

The system of two coupled equations for $\mathcal{C}^2$ and δ can be decoupled. Červený (2004) proposed a method, called semi-analytic, in which the equation for δ is fully separated from that for $\mathcal{C}^2$. Once δ has been found, $\mathcal{C}^2$ can be determined by an explicit formula. However, the price paid for decoupling is the more complicated non-polynomial form of the equation for δ. The problems with the non-physical solutions remain.

2.2 Componental specification of the slowness vector

Inserting the componental specification (10) of slowness vector $\mathbf{p}$ into (2), we obtain a system of three linear equations for the components U_i of the complex-valued polarization vector $\mathbf{U}$:

$$a_{ijkl}\left(\sigma n_j + p_j^{\Sigma}\right)\left(\sigma n_l + p_l^{\Sigma}\right)U_k = U_i, \quad i = 1, 2, 3. \tag{16}$$

The condition of solvability of system (16) reads

$$\det\left[a_{ijkl}\left(\sigma n_j + p_j^{\Sigma}\right)\left(\sigma n_l + p_l^{\Sigma}\right) - \delta_{ik}\right] = 0. \tag{17}$$

Assuming a_{ijkl}, $\mathbf{n}$ and $\mathbf{p}^{\Sigma}$ are known, eq. (17) represents an algebraic equation of the sixth degree in σ. The six roots σ of (17) correspond to *P*, *S1*, and *S2* inhomogeneous plane waves, propagating away from Σ, to both sides of it. One polarization vector $\mathbf{U}$ corresponds to each root σ. For a non-degenerate case, i.e. for a root which does not coincide with any other, the relevant complex-valued polarization vector $\mathbf{U}$ can be obtained from (16), supplemented by a suitable normalization condition, for example:

$$U_iU_i^* = 1, \quad \text{or} \quad U_iU_i = 1. \tag{18}$$

Eq. (17) is a basic equation for the determination of the slowness vectors of reflected/transmitted waves at structural interfaces (see, for example, Fedorov (1968), Gajewski & Pšenčík (1987) and Červený (2001, p. 527)). These references, however, treat only real-valued quantities a_{ijkl} (perfectly elastic media).

The two eqs (16) and (17) are sufficient for complete solution of the problem of homogeneous and inhomogeneous plane waves propagating in viscoelastic anisotropic media. Let us mention that the above problem can also be solved using an alternative approach based on 6×6 matrices and relevant eigenvalue problems. We emphasize that we mention this approach here only for completeness; it is not used in this article at all. It may be proved that the roots σ of (17) can be alternatively determined as eigenvalues of the 6×6 complex-valued matrix $\mathbf{\Pi}$:

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} \end{pmatrix}, \tag{19}$$

where the 3×3 complex-valued partition matrices $\mathbf{\Pi}_{IJ}$ read

$$\begin{aligned} \mathbf{\Pi}_{11} &= -\mathsf{C}^{(1)-1}\mathsf{C}^{(2)} = \mathbf{\Pi}_{22}^{\mathrm{T}}, \\ \mathbf{\Pi}_{12} &= -\mathsf{C}^{(1)-1}, \\ \mathbf{\Pi}_{21} &= -\mathbf{I}_3 + \mathsf{C}^{(4)} - \mathsf{C}^{(3)}\mathsf{C}^{(1)-1}\mathsf{C}^{(2)}, \\ \mathbf{\Pi}_{22} &= -\mathsf{C}^{(3)}\mathsf{C}^{(1)-1}, \end{aligned} \tag{20}$$

with

$$C_{ik}^{(1)} = a_{ijkl}n_jn_l, \quad C_{ik}^{(2)} = a_{ijkl}n_jp_l^{\Sigma}, \quad C_{ik}^{(3)} = a_{ijkl}p_j^{\Sigma}n_l, \quad C_{ik}^{(4)} = a_{ijkl}p_j^{\Sigma}p_l^{\Sigma}. \tag{21}$$

$\mathbf{I}_3$ in (20) is a 3×3 identity matrix. Eigenvalues σ are solutions of the characteristic equation

$$\det[\mathbf{\Pi} - \sigma\mathbf{I}_6] = 0, \tag{22}$$

where $\mathbf{I}_6$ is a 6×6 identity matrix.

The eigenvalue problem (22) for the 6×6 matrix $\mathbf{\Pi}$, given by (19)–(21), has been well known in seismology from the displacement–stress propagator matrix computations, and also in mechanics and applied mathematics, where it is usually called the Stroh formalism. For appropriate references see the Introduction.

Thus, if a real-valued unit vector **n**, perpendicular to plane Σ, and a complex-valued vector $\mathbf{p}^{\Sigma}$, situated in plane Σ, are given, the determination of the slowness vector **p** reduces to the solution of an algebraic equation of the sixth degree (17), or to the solution of a conventional eigenvalue problem (22) for a 6 × 6 complex-valued matrix **Π**.

In the componental specification, directions **N** of propagation vectors **P** of the individual waves are not known in advance and are not the same for all waves; they are obtained as a result of computation. If we wish to study the inhomogeneous plane waves propagating in the direction of an *a priori* specified unit vector **N**, we have to choose p_i^{Σ} in a special way (see the next section).

Once a value of σ has been found, we can determine the relevant real-valued propagation vector **P**, attenuation vector **A**, unit vectors **N** and **M**, phase velocity $\mathcal{C}$, attenuation–propagation ratio δ and the attenuation angle γ. They are given by the relations,

$$\begin{aligned}
\mathbf{P} &= \mathbf{n}(\mathrm{Re}\,\sigma) + \mathrm{Re}\,\mathbf{p}^{\Sigma},\\
|\mathbf{P}| &= [(\mathrm{Re}\,\sigma)^2 + (\mathrm{Re}\,\mathbf{p}^{\Sigma})(\mathrm{Re}\,\mathbf{p}^{\Sigma})]^{1/2},\\
\mathbf{A} &= \mathbf{n}(\mathrm{Im}\,\sigma) + \mathrm{Im}\,\mathbf{p}^{\Sigma},\\
|\mathbf{A}| &= [(\mathrm{Im}\,\sigma)^2 + (\mathrm{Im}\,\mathbf{p}^{\Sigma})(\mathrm{Im}\,\mathbf{p}^{\Sigma})]^{1/2},\\
\mathbf{N} &= \mathbf{P}/|\mathbf{P}|,\\
\mathbf{M} &= \mathbf{A}/|\mathbf{A}|,\\
\mathcal{C} &= 1/|\mathbf{P}|,\\
\delta &= |\mathbf{A}|/|\mathbf{P}|,\\
\cos\gamma &= [(\mathrm{Re}\,\mathbf{p}^{\Sigma})(\mathrm{Im}\,\mathbf{p}^{\Sigma}) + (\mathrm{Re}\,\sigma)(\mathrm{Im}\,\sigma)]/|\mathbf{P}||\mathbf{A}|. \qquad (23)
\end{aligned}$$

Inserting σ into (16) and solving system (16) with one of the normalization conditions (18), we can also find the relevant polarization vector **U**.

2.3 Mixed specification of the slowness vector

If we wish to study inhomogeneous plane waves with the wave front propagating in the direction of a known unit vector **N**, it is suitable to use the componental specification with plane Σ perpendicular to **N**. In this way, vector **N** actually specifies the unit vector **n** (or −**n**), perpendicular to Σ. As plane Σ represents a wave front in this case, we have $\mathrm{Re}(p_i^{\Sigma}) = 0$. Consequently, vector $\mathbf{p}^{\Sigma}$ is purely imaginary, $\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}$ (see eq. 11). Here **m** is a real-valued unit vector, parallel to the wave front (thus $\mathbf{n}\cdot\mathbf{m} = 0$), and D is an inhomogeneity parameter.

Using (11) in eqs (16) and (17), we obtain the basic relations for σ and **U** of the mixed specification of the slowness vector. For σ, we obtain the algebraic equation of the sixth degree:

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0, \qquad (24)$$

and for polarization vector **U** the system of linear equations

$$a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l)U_k = U_i, \quad i = 1, 2, 3. \qquad (25)$$

To determine polarization vector **U** from (25), a suitable normalization condition for **U** should be used (see eq. 18).

Let us again emphasize that a_{ijkl}, U_j, p_j and σ are generally complex-valued, whereas **n** and **m** are real-valued, mutually perpendicular, unit vectors, and D is a real-valued scalar.

Eq. (24) for σ simplifies considerably for homogeneous plane waves ($D = 0$). It yields

$$\det\left[a_{ijkl}n_j n_l - (1/\sigma)^2\delta_{ik}\right] = 0.$$

This immediately provides

$$(1/\sigma)^2 = G, \qquad (26)$$

where G is an eigenvalue of the 3 × 3 complex-valued Christoffel matrix $a_{ijkl}n_j n_l$. Eq. (26) is valid for homogeneous plane waves only, but otherwise it is quite universal, valid for any *P*, *S1* or *S2* homogeneous plane wave propagating in an anisotropic/isotropic, viscoelastic/perfectly elastic medium.

The two eqs (24) and (25) are sufficient for complete solution of the problem of homogeneous and inhomogeneous plane waves propagating in unbounded viscoelastic anisotropic media *in a specified direction* **N**. All analytical results obtained in this paper, and numerical results obtained in the related paper (Červený & Pšenčík 2005) are based on the solution of the algebraic equation of the sixth degree (24) for σ, and on the solution of the system of linear algebraic eqs (25) for U_i.

Only for completeness, we present here also the approach based on the solution of the eigenvalue problem (22) for the mixed specification. The equations for the 6 × 6 matrix **Π** remain the same as in the componental specification, only $\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}$ is used. Consequently, eqs (19) and (20) are not changed, but eqs (21) read

$$C_{ik}^{(1)} = a_{ijkl}n_j n_l, \quad C_{ik}^{(2)} = \mathrm{i}Da_{ijkl}n_j m_l, \quad C_{ik}^{(3)} = \mathrm{i}Da_{ijkl}m_j n_l, \quad C_{ik}^{(4)} = -D^2 a_{ijkl}m_j m_l. \qquad (27)$$

We emphasize again that the approach based on the eigenvalue problem (22) for the matrix $\mathbf{\Pi}$ is not used in this paper at all. Once the quantity σ has been found, we can determine the relevant real-valued propagation vector **P**, attenuation vector **A**, unit vectors **N** and **M**, phase velocity $\mathcal{C}$, attenuation–propagation ratio δ and attenuation angle γ:

$$\begin{aligned}
\mathbf{P} &= \mathbf{n}\,\mathrm{Re}\,\sigma,\\
|\mathbf{P}| &= |\mathrm{Re}\,\sigma|,\\
\mathbf{A} &= \mathbf{n}\,\mathrm{Im}\,\sigma + D\mathbf{m},\\
|\mathbf{A}| &= [(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2},\\
\mathbf{N} &= \mathbf{P}/|\mathbf{P}| = \epsilon\mathbf{n},\\
\mathbf{M} &= \mathbf{A}/|\mathbf{A}|,\\
\mathcal{C} &= 1/|\mathrm{Re}\,\sigma|,\\
\delta &= |\mathbf{A}|/|\mathbf{P}|,\\
\cos\gamma &= \epsilon\,\mathrm{Im}\,\sigma/[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2},
\end{aligned} \tag{28}$$

where

$$\epsilon = \mathrm{Re}\,\sigma/|\mathrm{Re}\,\sigma| = \pm 1. \tag{29}$$

The mixed specification offers a simple and general algorithm to study inhomogeneous or homogeneous plane waves propagating in the direction of a known unit vector **N** in unbounded, anisotropic or isotropic, viscoelastic or perfectly elastic media based on eqs (24) and (25). Assume that two real-valued, mutually perpendicular unit vectors $\mathbf{n} = \pm\mathbf{N}$ and **m** are known (**N** perpendicular to the wave front Σ and **m** tangential to it), and that the real-valued inhomogeneity parameter D is given. The determination of slowness vector **p** is then reduced to the solution of algebraic eq. (24) of the sixth degree. The algorithm is independent of the choice of sign in $\mathbf{n} = \pm\mathbf{N}$. The six roots σ_i ($i = 1, 2, \ldots, 6$) of (24) correspond to *P*, *S1*, and *S2* inhomogeneous plane waves, propagating to both sides of Σ. Three of them propagate in the direction of $\mathbf{N} = \mathbf{n}$, and other three in the opposite direction, $\mathbf{N} = -\mathbf{n}$. (See Červený & Pšenčík (2003) for numerical examples, and a detailed numerical treatment based on eq. (24) in Červený & Pšenčík (2005).)

The most important advantages of the mixed specification over the directional specification in the determination of the slowness vector of an inhomogeneous plane wave propagating in the direction of **N** are as follows:

(1) Conventional numerical algorithms can be used in the mixed specifications. Contrary to this, the directional specification requires the solution of a generally complicated system of two coupled equations.

(2) The mixed specification yields the slowness vector not only for $\mathbf{N} = \mathbf{n}$ but also for $\mathbf{N} = -\mathbf{n}$.

(3) Since eqs (28) yield real-valued, non-negative phase velocity $\mathcal{C}$ for any complex-valued σ, the mixed specification yields a physically acceptable solution for any choice of real-valued, mutually perpendicular, unit vectors $\mathbf{n} = \pm\mathbf{N}$ and **m** and any choice of real-valued inhomogeneity parameter D. It never yields non-physical solutions $\mathcal{C}^2 < 0$, as the computations in the **N**, **M** domain do.

Let us now briefly discuss the differences between the mixed and componental specifications. In fact, the mixed specification is a special case of the componental specification. Thus, eqs (16) and (17) can be also used in the mixed specification if **n** and $\mathbf{p}^{\Sigma}$ are properly specified. The advantage of the mixed specification consists only in a simpler way of parametrization of homogeneous and inhomogeneous plane waves propagating in unbounded viscoelastic anisotropic media in a *given direction of propagation* **N**. We can merely use $\mathbf{n} = \mathbf{N}$ in this case. The vector **n** and a unit vector **m** perpendicular to it then specify the propagation–attenuation plane, in which the complex-valued slowness vector **p** is situated. The inhomogeneity parameter D controls inhomogeneity of the plane wave under consideration. If the componental specification were used (with **n** and $\mathbf{p}^{\Sigma}$ given), the direction of propagation **N** would not be known in advance. It should be sought.

The componental specification is more useful in applications in which the directions of propagation **N** of the considered waves are not known in advance, and are to be determined (reflection/transmission problem, initial surface problem). In this article, however, we are not interested in these applications so that the mixed specification is most suitable.

It should be emphasized that the assumed mutual perpendicularity of **n** and **m** does not restrict the generality and universality of (12). A complete system of all possible homogeneous and inhomogeneous plane waves propagating in directions $\mathbf{N} = \mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$ is obtained by choosing all possible real-valued unit vectors **m**, perpendicular to **n**, and all possible inhomogeneity parameters D either from the interval $\langle 0, \infty)$ or from $(-\infty, 0\rangle$.

3 PLANE OF SYMMETRY OF AN ANISOTROPIC VISCOELASTIC MEDIUM OF MONOCLINIC SYMMETRY

In this section, we consider inhomogeneous plane waves propagating in the plane of symmetry Σ^S of a monoclinic viscoelastic medium. We choose the Cartesian coordinate system x_i in such a way that the plane of symmetry Σ^S corresponds to coordinate plane x_1x_3. In the Voigt notation, the density normalized elastic moduli $A_{\mu\nu}$ ($\mu, \nu = 1, 2, \ldots, 6$) are then given by the matrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & A_{15} & 0 \\ A_{12} & A_{22} & A_{23} & 0 & A_{25} & 0 \\ A_{13} & A_{23} & A_{33} & 0 & A_{35} & 0 \\ 0 & 0 & 0 & A_{44} & 0 & A_{46} \\ A_{15} & A_{25} & A_{35} & 0 & A_{55} & 0 \\ 0 & 0 & 0 & A_{46} & 0 & A_{66} \end{pmatrix} \tag{30}$$

(see Carcione 2001, eq. 1.37). For a viscoelastic monoclinic medium, $A_{\mu\nu}$ are complex-valued. As special cases of (30), we can consider an orthorhombic viscoelastic medium (with $A_{15} = A_{25} = A_{35} = A_{46} = 0$), a hexagonal viscoelastic medium with axis of symmetry along the x_3-axis ($A_{15} = A_{25} = A_{35} = A_{46} = 0$, and $A_{11} = A_{22}$, $A_{44} = A_{55}$, $A_{13} = A_{23}$, $A_{12} = A_{11} - 2A_{66}$), etc. We assume that matrix $\mathrm{Re}A_{\alpha\beta}$ is positive definite, and matrix $\mathrm{Im}A_{\alpha\beta}$ is negative definite, or zero.

We concentrate on plane waves propagating in the symmetry plane Σ^S, and consider

$$p_2 = 0. \tag{31}$$

Thus, in this case the plane of symmetry Σ^S corresponds to propagation-attenuation plane $\Sigma^{\parallel}$. The elements of the 3×3 generalized Christoffel matrix $\Gamma_{ik} = a_{ijkl}p_j p_l$ are given by relations

$$\begin{aligned} \Gamma_{11}(p_n) &= A_{11}p_1^2 + A_{55}p_3^2 + 2A_{15}p_1p_3, \\ \Gamma_{22}(p_n) &= A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3, \\ \Gamma_{33}(p_n) &= A_{55}p_1^2 + A_{33}p_3^2 + 2A_{35}p_1p_3, \\ \Gamma_{13}(p_n) &= \Gamma_{31}(p_n) = A_{15}p_1^2 + A_{35}p_3^2 + (A_{13} + A_{55})p_1p_3, \\ \Gamma_{12}(p_n) &= \Gamma_{21}(p_n) = \Gamma_{23}(p_n) = \Gamma_{32}(p_n) = 0. \end{aligned} \tag{32}$$

Constraint relation (3) factorizes:

$$\det[\Gamma_{ik}(p_n) - \delta_{ik}] = [\Gamma_{22}(p_n) - 1]\det\begin{bmatrix} \Gamma_{11}(p_n) - 1 & \Gamma_{13}(p_n) \\ \Gamma_{13}(p_n) & \Gamma_{33}(p_n) - 1 \end{bmatrix} = 0. \tag{33}$$

Eq. (33) can be satisfied in two cases:

(1) *Case of SH plane waves.* The constraint relation reads

$$\Gamma_{22}(p_n) - 1 = 0. \tag{34}$$

It follows from (2) that the polarization vector **U** is perpendicular to the plane of symmetry Σ^S,

$$\mathbf{U}(p_n) = (0, U_2(p_n), 0)^{\mathrm{T}}. \tag{35}$$

It is common to call the relevant plane wave the *SH* plane wave.

(2) *Case of P and SV waves.* The constraint relation reads

$$\det\begin{bmatrix} \Gamma_{11}(p_n) - 1 & \Gamma_{13}(p_n) \\ \Gamma_{13}(p_n) & \Gamma_{33}(p_n) - 1 \end{bmatrix} = 0. \tag{36}$$

The polarization vectors **U** of the relevant plane waves are situated in the plane of symmetry Σ^S,

$$\mathbf{U}(p_n) = (U_1(p_n), 0, U_3(p_n))^{\mathrm{T}}, \tag{37}$$

where $U_1(p_n)$ and $U_3(p_n)$ satisfy the system of two complex-valued linear equations resulting from (2),

$$\begin{aligned} (\Gamma_{11}(p_n) - 1)U_1 + \Gamma_{13}(p_n)U_3 &= 0, \\ \Gamma_{13}(p_n)U_1 + (\Gamma_{33}(p_n) - 1)U_3 &= 0. \end{aligned} \tag{38}$$

Eq. (38) should be supplemented by a proper normalization condition (see eq. 18).

In this section, we derive the complete equations for the *SH*, *P* and *SV* homogeneous and inhomogeneous plane waves, propagating in the symmetry plane Σ^S.

3.1 Directional specification of the slowness vector

We first treat the homogeneous and inhomogeneous *SH* plane waves, and after this briefly comment on the *P* and *SV* waves.

3.1.1 Case of SH waves

For *SH* waves, constraint relation (34), with (32) for $\Gamma_{22}(p_n)$, yields

$$A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 = 1. \tag{39}$$

Inserting directional specification $p_i = \mathcal{C}^{-1}(N_i + \mathrm{i}\delta M_i)$, and separating the real and imaginary parts, we obtain two equations for δ and $\mathcal{C}^2$:

$$a_1 + \delta a_2 - \delta^2 a_3 = \mathcal{C}^2, \tag{40}$$

$$-a_4 + \delta a_5 + \delta^2 a_6 = 0, \tag{41}$$

where

$$\begin{aligned} a_1 &= A^{\mathrm{R}}_{66} N_1^2 + A^{\mathrm{R}}_{44} N_3^2 + 2A^{\mathrm{R}}_{46} N_1 N_3, \\ a_2 &= -2\left[A^{\mathrm{I}}_{66} N_1 M_1 + A^{\mathrm{I}}_{44} N_3 M_3 + A^{\mathrm{I}}_{46}(M_1 N_3 + N_1 M_3)\right], \\ a_3 &= A^{\mathrm{R}}_{66} M_1^2 + A^{\mathrm{R}}_{44} M_3^2 + 2A^{\mathrm{R}}_{46} M_1 M_3, \\ a_4 &= -\left[A^{\mathrm{I}}_{66} N_1^2 + A^{\mathrm{I}}_{44} N_3^2 + 2A^{\mathrm{I}}_{46} N_1 N_3\right], \\ a_5 &= 2\left[A^{\mathrm{R}}_{66} N_1 M_1 + A^{\mathrm{R}}_{44} N_3 M_3 + A^{\mathrm{R}}_{46}(N_1 M_3 + M_1 N_3)\right], \\ a_6 &= -\left[A^{\mathrm{I}}_{66} M_1^2 + A^{\mathrm{I}}_{44} M_3^2 + 2A^{\mathrm{I}}_{46} M_1 M_3\right]. \end{aligned} \tag{42}$$

Here we have used the notation $A^{\mathrm{R}}_{ij} = \mathrm{Re}(A_{ij})$, $A^{\mathrm{I}}_{ij} = \mathrm{Im}(A_{ij})$.

For a given model of a viscoelastic monoclinic medium (30), and for given unit vectors $\mathbf{N} \equiv (N_1, 0, N_3)$ and $\mathbf{M} \equiv (M_1, 0, M_3)$, quantities a_i, $i = 1, 2, \ldots, 6$ in (42) are known. Thus, eqs (40) and (41) can be used to determine δ and $\mathcal{C}^2$.

An analogous system of equations for an *SH* inhomogeneous plane wave propagating in a viscoelastic transversely isotropic medium was first derived and numerically studied by Krebes & Le (1994): for *SH* inhomogeneous plane waves propagating in the symmetry plane of a monoclinic viscoelastic medium see Carcione & Cavallini (1995) and Carcione (2001); for electromagnetic waves see Carcione & Cavallini (1997).

A brief note on the coefficients a_i in (42). We have assumed that matrix $\mathrm{Re}A_{\alpha\beta}$ is positive definite, and matrix $\mathrm{Im}A_{\alpha\beta}$ negative definite or zero, where $A_{\alpha\beta}$ is given by (30). For *SH* waves, the 6×6 matrix $A_{\alpha\beta}$ reduces to the 2×2 matrix

$$\begin{pmatrix} A_{66} & A_{46} \\ A_{46} & A_{44} \end{pmatrix}. \tag{43}$$

Then a_1 and a_3 are positive and a_4 and a_6 non-negative, but a_2 and a_5 may be positive, zero or negative.

Eq. (41) for δ is quadratic, and for $a_6 \neq 0$ yields the solutions

$$\delta_{1,2} = -a_5/2a_6 \pm \sqrt{(a_5/2a_6)^2 + a_4/a_6}. \tag{44}$$

As the imaginary part of matrix (43) is negative definite, or zero, the expression under square root in (44) is non-negative, so that δ_1 and δ_2 are real-valued. As δ must be non-negative, we have to use the '+' sign in (44),

$$\delta = -a_5/2a_6 + \sqrt{(a_5/2a_6)^2 + a_4/a_6}. \tag{45}$$

Inserting this δ into (40) yields the square of phase velocity, $\mathcal{C}^2$.

Thus, for given directions $\mathbf{N}$ and $\mathbf{M}$ and for $a_6 \neq 0$, (45) and (40) represent the final equations for the determination of the attenuation–propagation ratio δ and phase velocity $\mathcal{C}$. The slowness vector $\mathbf{p}$ is then given by (9), and the attenuation angle γ by relation (8), $\cos\gamma = \mathbf{N} \cdot \mathbf{M}$.

If $a_6 = 0$, also $a_4 = 0$. This means that the medium is perfectly elastic. Eq. (41) then yields $a_5\delta = 0$. This is satisfied in two cases:

(a) $\delta = 0$, a_5 arbitrary: Note that $\delta = 0$ indicates the real-valued slowness vector. Thus, a regular plane wave propagating in a perfectly elastic medium is obtained. Its phase velocity is $\mathcal{C} = \sqrt{a_1}$ (see 40) for $\delta = 0$.

(b) $a_5 = 0$, δ arbitrary: This corresponds to an inhomogeneous plane wave propagating in a perfectly elastic medium. Condition $a_5 = 0$, however, may be then satisfied only for a special choice of $\mathbf{N}$ and $\mathbf{M}$. In isotropic media ($A_{44} = A_{66}$, $A_{46} = 0$), condition $a_5 = 0$ leads to the well-known condition of perpendicularity of $\mathbf{N}$ and $\mathbf{M}$. In anisotropic media, condition $a_5 = 0$ is satisfied for $\mathbf{N}$ and $\mathbf{M}$ making an angle different from 90^0 (but strictly defined).

There is, however, a problem in the determination of $\mathcal{C}^2$ and δ using (40) and (45). For certain combinations of $\mathbf{N}$ and $\mathbf{M}$, eqs (45) with (40) can yield $\mathcal{C}^2 \leq 0$. This interesting result was discovered independently by Krebes & Le (1994) and by Carcione & Cavallini (1995, 1997) (see also Carcione 2001). Let us consider a fixed attenuation angle γ ($\cos\gamma = \mathbf{N} \cdot \mathbf{M}$), and unit propagation vector $\mathbf{N}$ varying. Carcione (2001, Section 4.4.6) calls $\mathbf{N}$ for which $\mathcal{C}^2 \leq 0$ *forbidden directions of propagation*. Carcione & Cavallini (1995) also speak of 'stop bands', where there is no wave propagation. Such forbidden directions of propagation do not arise if the componental or mixed specification is used to parametrize inhomogeneous plane waves. For a detailed explanation see Červený & Pšenčík (2005).

3.1.2 P and SV inhomogeneous plane waves

For *P* and *SV* inhomogeneous plane waves the constraint relation is given by (36), with $\Gamma_{11}(p_n)$, $\Gamma_{33}(p_n)$ and $\Gamma_{13}(p_n)$ given by (32). The two relevant coupled equations for δ and $\mathcal{C}^2$ are not simple to solve. Moreover, the non-physical solutions ($\mathcal{C}^2 \leq 0$) may also be obtained. For this reason, we do not discuss this case here. The method based on the mixed specification of the slowness vector (see Section 3.3) yields a simpler algorithm and avoids the problem of non-physical solutions completely.

3.2 Componental specification of the slowness vector

We now apply componental specification (10) to inhomogeneous and homogeneous plane waves propagating in the plane of symmetry Σ^S of a monoclinic viscoelastic medium (see eq. 30). We assume that the unit normal **n** to plane Σ is situated in the plane of symmetry Σ^S, so that $n_2 = 0$. We also assume $p_2^\Sigma = 0$.

We consider first *SH* plane waves, and then *SV* and *P* plane waves. The results may find applications in the solution of reflection/transmission problems of inhomogeneous plane waves and in initial-value problems, in which the initial values of slowness vector, $\mathbf{p}^\Sigma$, are given along an initial plane Σ.

3.2.1 Case of SH waves

Inserting the componental specification $p_i = \sigma n_i + p_i^\Sigma$ into the constraint relation (34) for *SH* waves yields

$$A_{44}\left(\sigma n_3 + p_3^\Sigma\right)^2 + A_{66}\left(\sigma n_1 + p_1^\Sigma\right)^2 + 2A_{46}\left(\sigma n_1 + p_1^\Sigma\right)\left(\sigma n_3 + p_3^\Sigma\right) = 1. \tag{46}$$

This is a quadratic equation in σ,

$$\sigma^2\Gamma_{22} + 2\sigma E_{22} + F_{22} - 1 = 0, \tag{47}$$

where Γ_{22}, E_{22} and F_{22} are given by the relations (see 32)

$$\begin{aligned}\Gamma_{22} &= A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3,\\ E_{22} &= A_{66}n_1p_1^\Sigma + A_{44}n_3p_3^\Sigma + A_{46}\left(n_1p_3^\Sigma + n_3p_1^\Sigma\right),\\ F_{22} &= A_{66}p_1^{\Sigma 2} + A_{44}p_3^{\Sigma 2} + 2A_{46}p_1^\Sigma p_3^\Sigma.\end{aligned} \tag{48}$$

The two solutions σ are as follows,

$$\sigma_{1,2} = \left(-E_{22} \pm \sqrt{\Gamma_{22} + E_{22}^2 - \Gamma_{22}F_{22}}\right)/\Gamma_{22}. \tag{49}$$

Using (48) and (32), eq. (49) can be expressed in a simpler form,

$$\sigma_{1,2} = -(E_{22}/\Gamma_{22}) \pm \sqrt{1/\Gamma_{22} - \left(p_1^\Sigma n_3 - n_1p_3^\Sigma\right)^2\Delta/\Gamma_{22}^2}, \tag{50}$$

where the quantity Δ is given by the relation

$$\Delta = A_{44}A_{66} - A_{46}^2. \tag{51}$$

3.2.2 Case of SV and P waves

Inserting $p_i = \sigma n_i + p_i^\Sigma$ into constraint relation (36) yields an algebraic equation of the fourth degree in σ,

$$\sigma^4 a_1 + 2\sigma^3 b_1 + \sigma^2(4a_2 + b_2 - c_1) + 2\sigma(b_3 - c_2) + a_3 - c_3 + 1 = 0, \tag{52}$$

where

$$\begin{aligned}&a_1 = \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2, \quad b_1 = \Gamma_{11}E_{33} + F_{33}E_{11} - 2\Gamma_{13}E_{13}, \quad c_1 = \Gamma_{11} + \Gamma_{33},\\ &a_2 = E_{11}E_{33} - E_{13}^2, \quad b_2 = \Gamma_{11}F_{33} + \Gamma_{33}F_{11} - 2\Gamma_{13}F_{13}, \quad c_2 = E_{11} + E_{33},\\ &a_3 = F_{11}F_{33} - F_{13}^2, \quad b_3 = F_{11}E_{33} + F_{33}E_{11} - 2F_{13}E_{13}, \quad c_3 = F_{11} + F_{33}.\end{aligned} \tag{53}$$

We have also used the notation

$$\Gamma_{ij} = \Gamma_{ij}(\mathbf{n}), \quad F_{ij} = \Gamma_{ij}(\mathbf{p}^\Sigma), \tag{54}$$

where $\Gamma_{ij}(\mathbf{n})$ and $\Gamma_{ij}(\mathbf{p}^\Sigma)$ are given by (32), with **p** substituted by **n** and $\mathbf{p}^\Sigma$. Finally, quantities E_{ij} are given by relations

$$\begin{aligned}E_{11} &= A_{11}n_1p_1^\Sigma + A_{55}n_3p_3^\Sigma + A_{15}\left(n_1p_3^\Sigma + n_3p_1^\Sigma\right),\\ E_{33} &= A_{55}n_1p_1^\Sigma + A_{33}n_3p_3^\Sigma + A_{35}\left(n_1p_3^\Sigma + n_3p_1^\Sigma\right),\\ E_{13} &= A_{15}n_1p_1^\Sigma + A_{35}n_3p_3^\Sigma + \frac{1}{2}(A_{13} + A_{55})\left(n_1p_3^\Sigma + n_3p_1^\Sigma\right).\end{aligned} \tag{55}$$

If the density normalized viscoelastic moduli A_{11}, A_{13}, A_{15}, A_{33}, A_{35} and A_{55}, unit vector **n** and vector $\mathbf{p}^\Sigma$ are known, all coefficients of eq. (52) can be determined, and (52) can be solved for σ.

Inserting solution of (47) or (52) into (23), we obtain the relevant expressions for propagation vector **P**, attenuation vector **A**, unit vectors **N** and **M**, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ and attenuation angle γ.

3.3 Mixed specification of the slowness vector

The mixed specification may be treated as a special case of the componental specification discussed in Section 3.2, if we put $\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}$ (see 11). Consequently, for *SH*, *P* and *SV* plane waves propagating in the plane of symmetry Σ^{S} of a monoclinic viscoelastic medium we can use the equations of the previous section, where we insert $\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}$. Here $\mathbf{m}$ is a unit vector perpendicular to $\mathbf{n}$, and D is the inhomogeneity parameter.

As $\mathbf{m}$ is assumed to be perpendicular to $\mathbf{n} = (n_1, 0, n_3)$ and situated in the symmetry plane $\Sigma^{\parallel}$, we use $\mathbf{m}$ in the following form:

$$\mathbf{m} = (n_3, 0, -n_1)^{\mathrm{T}}. \tag{56}$$

Due to this choice of the unit vector $\mathbf{m}$, the propagation–attenuation plane coincides with the symmetry plane.

3.3.1 Case of SH waves

Inserting (11) and (56) into (50), we obtain simple explicit formulae for σ

$$\sigma_{1,2} = -\mathrm{i}D\Lambda/\Gamma_{22} \pm \left(1/\Gamma_{22} + D^2\Delta/\Gamma_{22}^2\right)^{1/2}, \tag{57}$$

where

$$\begin{aligned}\Gamma_{22} &= A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3,\\ \Lambda &= (A_{66} - A_{44})n_1n_3 + A_{46}\left(n_3^2 - n_1^2\right),\end{aligned} \tag{58}$$

and where Δ is given by (51).

3.3.2 Case of P and SV waves

In this case, we obtain a quartic equation for σ. Inserting (11) and (56) into (52) yields

$$\sigma^4 A_1 + 2\mathrm{i}D\sigma^3 B_1 - \sigma^2\left[D^2(4A_2 + B_2) + C_1\right] + 2\mathrm{i}\sigma\left(D^3B_3 + DC_2\right) + D^4A_3 + D^2C_3 + 1 = 0. \tag{59}$$

Here A_i, B_i and C_i ($i = 1, 2, 3$) are given by the same expressions as a_i, b_i and c_i (see 53), only $\mathbf{p}^{\Sigma}$ in E_{ij} and F_{ij} is substituted by $\mathbf{m}$. Standard methods for solving quartic equations can be used to determine the four roots σ of (59).

Once the values of σ have been found, we can use (28) and determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation–propagation ratio δ and attenuation angle γ, related to the *SH*, *SV* and *P* homogeneous or inhomogeneous plane wave under consideration:

$$\begin{aligned}P_1 &= n_1(\mathrm{Re}\,\sigma),\\ P_3 &= n_3(\mathrm{Re}\,\sigma),\\ A_1 &= n_1(\mathrm{Im}\,\sigma) + n_3D,\\ A_3 &= n_3(\mathrm{Im}\,\sigma) - n_1D,\\ |\mathbf{P}| &= |\mathrm{Re}\,\sigma|,\\ |\mathbf{A}| &= [(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2},\\ N_1 &= \epsilon n_1,\\ N_3 &= \epsilon n_3,\\ M_1 &= [n_1(\mathrm{Im}\,\sigma) + n_3D]/[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2},\\ M_3 &= [n_3(\mathrm{Im}\,\sigma) - n_1D]/[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2},\\ \mathcal{C} &= 1/|\mathrm{Re}\,\sigma|,\\ \delta &= [(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2}/|\mathrm{Re}\,\sigma|,\\ \cos\gamma &= \epsilon\,\mathrm{Im}\,\sigma/[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2}.\end{aligned} \tag{60}$$

Here ϵ is given by (29). We also get $P_2 = 0$, $A_2 = 0$, $N_2 = 0$, $M_2 = 0$.

3.4 Some properties of plane *SH* waves

The mixed specification of slowness vector $\mathbf{p}$, particularly eqs (57), (58) and (60), offers a simple possibility to investigate the properties of homogeneous and inhomogeneous plane *SH* waves, propagating in the symmetry planes of viscoelastic anisotropic media. Actually, this is the only case of inhomogeneous plane waves propagating in anisotropic viscoelastic media which can be investigated analytically. We present here only the results which follow from simple analytical considerations. For a detailed numerical treatment see Červený & Pšenčík (2005).

3.4.1 Direction of propagation **N**

As we can see from (60), $\mathbf{N} = \epsilon\mathbf{n}$, where ϵ equals either +1, or −1. Thus, the unit vector **N** specifying the direction of propagation vector **P** (perpendicular to the wave front) equals **n**, or is opposite to it. For $\mathrm{Re}\,\sigma > 0$, we obtain $\mathbf{N} = \mathbf{n}$, and for $\mathrm{Re}\,\sigma < 0$ $\mathbf{N} = -\mathbf{n}$.

3.4.2 Attenuation angle γ

The attenuation angle γ is introduced here as a positive angle between unit vectors **N** and **M**, $0^\circ \le \gamma \le 180^\circ$, in such a way that $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$. In the planar case that we are treating it has been common to use an 'oriented attenuation angle γ' in the seismological literature (although the word 'oriented' has been considered only tacitly). The oriented attenuation angle γ would then be uniquely defined in the range $-180^\circ < \gamma < 180^\circ$. Simply speaking, the sign of γ specifies to which side of the propagation vector **P** the attenuation vector **A** is pointing. Such a definition of γ has been particularly useful in the directional specification of the slowness vector, where γ is used as an input parameter of the inhomogeneous plane wave under consideration. In the mixed specification, however, we use D as the input parameter, and the correct orientation of **A** is fully specified by the sign of D. Thus, we do not need to consider an oriented attenuation angle γ at all. We always define γ by (60), in the range $0^\circ \le \gamma \le 180^\circ$. For $0 \le \cos\gamma \le 1$, we obtain $0^\circ \le \gamma \le 90^\circ$, and for $-1 \le \cos\gamma \le 0$, we get $90^\circ \le \gamma \le 180^\circ$.

3.4.3 Roots σ_1 *and* σ_2

It is reasonable to assume that the following relations are valid for any D:

$$\mathrm{Re}\,\sigma_1 > 0, \quad \mathrm{Re}\,\sigma_2 < 0. \tag{61}$$

Consequently, root $\sigma = \sigma_1$ corresponds to $\mathbf{N} = \mathbf{n}$ (the wave front propagating along **n**), and root $\sigma = \sigma_2$ corresponds to $\mathbf{N} = -\mathbf{n}$ (the wave front propagating against **n**). For homogeneous plane waves ($D = 0$), the validity of (61) is obvious from (57). Since the validity of (61) is not required anywhere in this paper, we do not give its proof here for a general case.

3.4.4 Exponential decay of amplitudes. Attenuation $|\mathbf{A}|$

The maximum exponential decay of amplitudes occurs along the attenuation vector **A**. In our treatment, however, we decompose the attenuation vector **A** into its two Cartesian components. The first component is along the direction of propagation **N** and the second along the wave front. The exponential decay (growth) along **N** is controlled by the quantity $\mathrm{Im}\sigma$, and the exponential decay along the wave front by D. For homogeneous plane waves, the exponential decay is only along **N**. Contrary to this, the decay is only along the wave front if $\mathrm{Im}\,\sigma = 0$. The complete attenuation effect is given by the modulus $|\mathbf{A}|$ of attenuation vector **A**, $|\mathbf{A}| = [(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2}$ (see 60). $|\mathbf{A}|$ is briefly referred to as the *attenuation*.

3.4.5 Homogeneous SH plane waves

For homogeneous plane waves, $\mathbf{N} \equiv \mathbf{M}$, and the inhomogeneity parameter D vanishes. Then (57) yields,

$$\sigma_{1,2} = \pm\sqrt{1/\Gamma_{22}}, \tag{62}$$

where Γ_{22} is given by (58). Eqs (62), (58) and (60) yield simple expressions for phase velocity $\mathcal{C}$, attenuation $|\mathbf{A}|$ and the attenuation–propagation ratio δ:

$$\mathcal{C} = 1/|\mathrm{Re}\sqrt{1/\Gamma_{22}}|, \quad |\mathbf{A}| = |\mathrm{Im}\sqrt{1/\Gamma_{22}}|, \quad \delta = |\mathrm{Im}\sqrt{1/\Gamma_{22}}|/|\mathrm{Re}\sqrt{1/\Gamma_{22}}|. \tag{63}$$

3.4.6 Isotropic viscoelastic media

The properties of inhomogeneous plane waves propagating in isotropic viscoelastic media have been extensively studied in the seismological literature (see the references given in the Introduction). Most of these studies, however, use the attenuation angle γ as the basic parameter of the inhomogeneous plane wave. It is interesting to see how the results change if the mixed specification of the slowness vector with inhomogeneity parameter D is used.

For isotropic viscoelastic media, the values of σ follow immediately from (57), if we use $A_{44} = A_{66}$ and $A_{46} = 0$. We then obtain a surprisingly simple result:

$$\sigma_{1,2} = \pm\left(1/A_{44} + D^2\right)^{1/2}. \tag{64}$$

The quantity σ is applicable to inhomogeneous plane S waves propagating in any 3-D medium, in any direction. It is invariant with respect to unit vectors **n** and **m**. If we use a standard notation $A_{44} = V_S^2(1 - \mathrm{i}/Q_S)$, where V_S is the real-valued velocity and Q_S the real-valued quality factor of S waves, we obtain,

$$\sigma_{1,2} = \pm\left[1/V_S^2(1 - \mathrm{i}/Q_S) + D^2\right]^{1/2}. \tag{65}$$

Let us summarize the properties of homogeneous and inhomogeneous plane *S* waves propagating in isotropic viscoelastic media, which follow from eqs (64) and (65):

(1) The values of σ and of all consequent quantities are direction independent, and do not depend on the sign of D (they are symmetrical with respect to $D = 0$).

(2) The maximum phase velocity is always obtained for homogeneous plane waves ($D = 0$), and equals $1/|\mathrm{Re}\sqrt{1/A_{44}}|$. With increasing $|D|$ the phase velocity decreases as $1/|\mathrm{Re}\sqrt{1/A_{44} + D^2}|$, and for $|D| \to \infty$ it approaches zero.

(3) The minimum attenuation is always obtained for homogeneous plane waves ($D = 0$). With increasing $|D|$, attenuation $|\mathbf{A}|$ increases.

(4) The attenuation angle γ is zero for homogeneous plane waves ($D = 0$), and increases to 90° with $|D|$ increasing. The boundary attenuation angle is always 90°.

Eqs (64) and (65) remain valid even for inhomogeneous plane *S* waves propagating in perfectly elastic isotropic media; we only insert $1/Q_S = 0$ or $\mathrm{Im}(A_{44}) = 0$. Then $\mathcal{C} = \mathcal{C}_{1,2} = 1/\sqrt{1/V_S^2 + D^2}$ and $\mathrm{Im}\,\sigma = 0$. The attenuation angle γ depends neither on direction nor on D, and equals 90°. For $D = 0$, the expression (60) for $\cos\gamma$ is indefinite, but both limits ($D \to 0$ and $-D \to 0$) yield $\gamma = 90^\circ$.

3.4.7 *Weakly inhomogeneous plane waves*

For small D, the square root in (57) may be expanded in terms of D. Then we obtain approximately:

$$\mathrm{Re}\,\sigma_{1,2} \doteq \pm a + bD \pm cD^2, \quad \mathrm{Im}\,\sigma_{1,2} \doteq \pm e - fD \pm gD^2. \tag{66}$$

Here

$$a = \mathrm{Re}(1/\Gamma_{22})^{1/2}, \quad b = \mathrm{Im}(\Lambda/\Gamma_{22}), \quad c = \frac{1}{2}\mathrm{Re}\big(\Delta/\Gamma_{22}^{3/2}\big), \quad e = \mathrm{Im}(1/\Gamma_{22})^{1/2}, \quad f = \mathrm{Re}(\Lambda/\Gamma_{22}), \quad g = \frac{1}{2}\mathrm{Im}\big(\Delta/\Gamma_{22}^{3/2}\big). \tag{67}$$

The coefficient g is usually very small. Thus, for small D, $\mathrm{Re}\,\sigma_{1,2}$ can be approximated by a quadratic parabola and $\mathrm{Im}\,\sigma_{1,2}$ by a straight line. To simplify the notation, we consider here only one root of σ, namely $\sigma = \sigma_1$. The minimum of the parabola corresponds to $D = D^M$, where

$$D^M \doteq -b/2c. \tag{68}$$

As phase velocity $\mathcal{C}$ is given by the relation $\mathcal{C} = 1/|\mathrm{Re}\,\sigma|$, the phase velocity $\mathcal{C}^M$ corresponding to D^M is:

$$\mathcal{C}^M \doteq 1/(a - b^2/4c). \tag{69}$$

Thus, the maximum phase velocity (as a function of D) does not correspond to the homogeneous plane wave ($D = 0$), but to the weakly inhomogeneous plane wave with inhomogeneity parameter $D = D^M$, given by (68).

Similarly, we can determine the value of $D = D_0$, at which $\mathrm{Im}\,\sigma$ equals zero:

$$D_0 \doteq e/f. \tag{70}$$

For $D = D_0$, the attenuation angle γ equals 90°. Moreover, the value of D_0 is a boundary between the values of D for which amplitudes of inhomogeneous plane waves exponentially decay or grow along the direction of propagation $\mathbf{N}$.

Finally, attenuation $|\mathbf{A}|$ is minimum for the inhomogeneity parameter $D = D_{\mathrm{att}}$, given by the relation

$$D_{\mathrm{att}} \doteq ef/(1 + f^2). \tag{71}$$

The attenuation $|\mathbf{A}|$ at $D = D_{\mathrm{att}}$ is given by the relation

$$A_{\mathrm{min}} \doteq e/\sqrt{1 + f^2}, \tag{72}$$

and is here referred to as the minimum attenuation. Thus, the minimum attenuation (as a function of D) does not correspond to the homogeneous plane wave ($D = 0$) but to the weakly inhomogeneous plane wave with the inhomogeneity parameter $D = D_{\mathrm{att}}$, given by (71).

The inhomogeneity parameters D^M and D_{att} also play an important role in the investigation of the energy flux of weakly inhomogeneous *SH* plane waves propagating in symmetry planes of an anisotropic viscoelastic medium. Although we do not discuss the energy flux in this paper, we present here (without derivation) two interesting properties valid for $D = D^M$ and $D = D_{\mathrm{att}}$. We introduce the complex-valued Poynting vector $\mathbf{F}$, and the time-averaged, real-valued energy flux $\mathbf{S} = \mathrm{Re}\,\mathbf{F}$. These quantities can be simply computed once the complex-valued slowness vector $\mathbf{p}$ is known. Then, for $D = D_{\mathrm{att}}$, the energy flux $\mathbf{S}$ and attenuation vector $\mathbf{A}$ are parallel. Similarly, for $D = D^M$, the real and imaginary parts of the Poynting vector, $\mathrm{Re}\,\mathbf{F}$ and $\mathrm{Im}\,\mathbf{F}$, are parallel.

3.4.8 *Strongly inhomogeneous SH plane waves*

For large $|D|$, eqs (57) yield approximately,

$$\mathrm{Re}\,\sigma_{1,2} \doteq D\,\mathrm{Im}(\Lambda/\Gamma_{22}) \pm |D|\mathrm{Re}\big(\Delta/\Gamma_{22}^2\big)^{1/2}, \quad \mathrm{Im}\,\sigma_{1,2} \doteq -D\,\mathrm{Re}(\Lambda/\Gamma_{22}) \pm |D|\,\mathrm{Im}\big(\Delta/\Gamma_{22}^2\big)^{1/2}. \tag{73}$$

We denote all quantities corresponding to the limiting case of infinite $|D|$ by asterisks. For $|D| \to \infty$, the first equation of (73) with (60) immediately yields

$$\mathcal{C}_{1,2}^{*} = 0. \tag{74}$$

Of course, the zero phase velocity $\mathcal{C}^* = 0$ is not realistic; it corresponds to the limiting case of $|D| \to \infty$, which cannot, in fact, occur. This limiting case, however, plays an important role in the discussion of inhomogeneous plane waves. Similarly, the second equation yields the expressions for the limiting (boundary) attenuation angles $\gamma^*_{1,2}$:

$$\cos\gamma^*_{1,2} = \theta_{1,2}/\left(\theta^2_{1,2} + 1\right)^{1/2}, \tag{75}$$

where $\theta_{1,2}$ are given by the relation

$$\theta_{1,2} = \mathrm{Im}\left(\Delta/\Gamma^2_{22}\right)^{1/2} \mp \mathrm{sgn}D\,\mathrm{Re}(\Lambda/\Gamma_{22}). \tag{76}$$

Thus, the boundary attenuation angle γ^* does not depend on the magnitude of D but only on the sign of D.

It was shown in Section 3.4.6 that the boundary attenuation angle always equals 90° in viscoelastic isotropic media. For viscoelastic anisotropic media, however, it usually differs from 90°, and depends on the unit vectors $\mathbf{n}$, $\mathbf{m}$ and on sgn D. For weakly dissipative media, it is usually close to 90° but may deviate considerably from 90° for strongly dissipative media. It may be both less than and greater than 90°.

3.4.9 *Perfectly elastic anisotropic media*

The density normalized moduli A_{44}, A_{66} and A_{46} are real-valued in this case. Consequently, Γ_{22}, Λ and Δ are also real-valued, and (57) yields

$$\mathrm{Re}\,\sigma_{1,2} = \pm\left(1 + D^2\Delta/\Gamma_{22}\right)^{1/2}/\Gamma^{1/2}_{22}, \quad \mathrm{Im}\,\sigma_{1,2} = -D\Lambda/\Gamma_{22}. \tag{77}$$

Consequently, phase velocity $\mathcal{C}$ and attenuation angle γ are given by the relations

$$\mathcal{C} = \mathcal{C}_{1,2} = \Gamma^{1/2}_{22}/\left(1 + D^2\Delta/\Gamma_{22}\right)^{1/2}, \quad \cos\gamma_{1,2} = \mp(D/|D|)\Lambda/\left(\Gamma^2_{22} + \Lambda^2\right)^{1/2}. \tag{78}$$

Thus the attenuation angle γ does not depend on D but only on the sign of D. If the medium is anisotropic and $\Lambda \neq 0$, the attenuation angle γ is different from 90°.

3.4.10 *Polarization of SH waves*

Both homogeneous and inhomogeneous *SH* plane waves are always linearly polarized in the direction perpendicular to Σ^S (see 35).

3.5 Some properties of plane *P* and *SV* waves

The properties of the slowness vectors of plane *P* and *SV* waves propagating in the plane of symmetry of a monoclinic viscoelastic medium are very similar to those of *SH* inhomogeneous plane waves discussed in the preceding section. It is, however, usually more difficult to find simple analytical estimates for the individual quantities, as the case of *P* and *SV* inhomogeneous plane waves requires the solution of the quartic eq. (59). Mostly, the numerical treatment is more effective (see Červený & Pšenčík 2005).

In two cases, however, the quartic eq. (59) in σ reduces to a quadratic equation in σ^2. This applies to homogeneous *P* and *SV* waves propagating in a symmetry plane of an anisotropic viscoelastic medium, and to *P* and *SV* inhomogeneous plane waves propagating in an isotropic viscoelastic medium. We present explicit solutions for both cases, as perturbation methods may be used to extend them to *P* and *SV* weakly inhomogeneous plane waves (small D) propagating in viscoelastic anisotropic media. We also briefly discuss, more or less qualitatively, some more general cases, for example strongly inhomogeneous plane waves.

We do not discuss here again the points made in Sections 3.4.1–3.4.4 derived for *SH* waves. They remain valid even for *P* and *SV* waves.

3.5.1 *Homogeneous plane P and SV waves*

For $D = 0$, the quartic eq. (59) in σ reduces to the quadratic equation in σ^2:

$$\sigma^4 A_1 - \sigma^2 C_1 + 1 = 0, \tag{79}$$

where $A_1 = \Gamma_{11}\Gamma_{33} - \Gamma^2_{13}$ and $C_1 = \Gamma_{11} + \Gamma_{33}$ are complex-valued coefficients. Consequently, the roots σ^2 for *P* and *SV* waves can be expressed analytically,

$$\sigma^2 = \frac{1}{2}\left(\Gamma_{11} + \Gamma_{33} \mp \sqrt{(\Gamma_{11} - \Gamma_{33})^2 + 4\Gamma^2_{13}}\right)\Big/\left(\Gamma_{11}\Gamma_{33} - \Gamma^2_{13}\right). \tag{80}$$

Here the '−' sign is for *P* waves and the '+' sign for *SV* waves. Each σ^2 in (80) yields two solutions σ, one to each side of Σ, both for *P* and *SV* waves.

We can also verify that the general relation $\sigma^2 = 1/G$, given in (26), remains valid. Here G is an eigenvalue of the complex-valued Christoffel matrix, corresponding to a *P* or *SV* wave, and satisfying the quadratic equation $G^2 - C_1 G + A_1 = 0$.

3.5.2 Plane P and SV waves in isotropic viscoelastic media

In isotropic media, B_1, C_2 and B_3 in (59) vanish. Thus, the quartic equation in σ reduces to a quadratic equation in σ^2. If we further use $A_1 = A_3 = A_{11}A_{55}$, $A_2 = C_1 = C_3 = A_{11} + A_{55}$ and $B_2 = A_{11}^2 + A_{55}^2$, valid for isotropic media, the quadratic equation reads

$$\sigma^4 A_{11}A_{55} - \sigma^2\big(2D^2 A_{11}A_{55} + A_{11} + A_{55}\big) + D^4 A_{11}A_{55} + D^2(A_{11} + A_{55}) + 1 = 0. \qquad (81)$$

Solving this equation, we obtain two simple solutions for σ^2

$$\sigma^2 = 1/A_{11} + D^2, \quad \text{for } P \text{ waves,}$$
$$\sigma^2 = 1/A_{55} + D^2, \quad \text{for } SV \text{ waves.} \qquad (82)$$

Thus, for isotropic viscoelastic media the expressions for σ^2 for plane *P* and *SV* waves have exactly the same form as for *SH* waves (see 64). Actually, this was expected. For *SH* and *SV* waves the expressions are identical, as $A_{44} = A_{55}$, and for *P* waves we use A_{11} instead of A_{44} (or A_{55}). Eq. (65) can also be used for all the three waves, only for *P* waves we replace V_S and Q_S by V_P and Q_P. The same conclusions (1)–(4) listed in Section 3.4.6 for *SH* waves also apply to *P* and *SV* waves. In addition, we obtain one interesting property of phase velocities of *P* and *S* waves. We take into account that $\sigma^2 \to D^2$ for increasing $|D|$. This implies that the differences between the phase velocities of *P* and *S* waves decrease with increasing *D*.

3.5.3 Weakly inhomogeneous plane P and SV waves

In a similar way as for *SH* waves, we can derive approximate relations for σ, assuming that the inhomogeneity parameter *D* is small. We put

$$\sigma \doteq \sigma_0 + \nu D, \qquad (83)$$

where σ_0 is given by (80), and corresponds to homogeneous plane waves. Using (59), we obtain an approximate relation for ν, valid for *D* small,

$$\nu \doteq -\mathrm{i}\big(\sigma_0^2 B_1 - C_2\big)\big/\big(2\sigma_0^2 A_1 - C_1\big). \qquad (84)$$

It follows from (83) and (84) that $\nu \doteq 0$ and $\sigma(D) \doteq \sigma(-D)$ for isotropic viscoelastic media (where $B_1 = C_2 = 0$) but not for anisotropic viscoelastic media. For weakly dissipative anisotropic media, ν is approximately purely imaginary. Consequently, for small *D* the differences between $\sigma(D)$ and $\sigma(-D)$ are particularly expressed in $\mathrm{Im}\,\sigma$ and in related quantities ($\cos\gamma$). For small *D*, the quantity $\mathrm{Im}\,\sigma$ as a function of *D* can be roughly approximated by an inclined straight line $\mathrm{Im}\,\sigma \doteq \mathrm{Im}\,\sigma_0 + D\,\mathrm{Im}\,\nu$, passing through $\mathrm{Im}\,\sigma_0$ for $D = 0$, and intersecting the axis $\mathrm{Im}\,\sigma = 0$ not at $D = 0$, as in isotropic media, but at $D = D_0$, where D_0 is given by the expression

$$D_0 \doteq -\mathrm{Im}\,\sigma_0/\mathrm{Im}\,\nu. \qquad (85)$$

An important property of D_0 is that attenuation angle γ equals 90° for $D = D_0$. This D_0 also represents the boundary between the values of *D* for which the amplitudes exponentially decay and exponentially grow along the unit propagation vector **N**.

The behaviour of $\mathrm{Re}\,\sigma$ for *D* small is similar, $\mathrm{Re}\,\sigma \doteq \mathrm{Re}\,\sigma_0 + D\,\mathrm{Re}\,\nu$. The values of $\mathrm{Re}\,\nu$ are, however, considerably smaller in magnitude than those of $\mathrm{Im}\,\nu$, but still exist. Consequently, phase velocity $\mathcal{C}$ (as a function of *D*) of a weakly inhomogeneous *P* or *SV* wave is not maximum for homogeneous plane waves ($D = 0$) but for a certain weakly inhomogeneous plane wave ($D = D^M \neq 0$). The same applies to the minimum of attenuation $|\mathbf{A}|$. The derivation of the relevant approximate analytical estimates is straightforward, but is not given here.

3.5.4 Strongly inhomogeneous P and SV waves

Here we consider large $|D|$. We introduce a new variable w instead of σ by a simple relation

$$\sigma = wD. \qquad (86)$$

Inserting (86) into (59), we obtain a quartic equation in w,

$$A_1 w^4 + 2\mathrm{i}w^3 B_1 - w^2\big[4A_2 + B_2 + D^{-2}C_1\big] - 2\mathrm{i}w\big(B_3 + D^{-2}C_2\big) + A_3 + D^{-2}C_3 + D^{-4} = 0. \qquad (87)$$

In the limiting case of $|D| \to \infty$, eq. (87) simplifies, but still remains quartic,

$$A_1 w^4 + 2\mathrm{i}w^3 B_1 - w^2(4A_2 + B_2) - 2\mathrm{i}wB_3 + A_3 = 0. \qquad (88)$$

For viscoelastic isotropic media we have $B_1 = B_3 = 0$ and $A_1 = A_3 = \frac{1}{2}(4A_2 + B_2) = A_{11}A_{22}$. Eq. (88) then simplifies to $w^4 - 2w^2 + 1 = 0$, which has a double root $w^2 = 1$, and yields $\sigma^2 = D^2$ for $|D| \to \infty$. For anisotropic viscoelastic media, the solution σ for *P* and *SV* waves may be expressed for $|D| \to \infty$ in the following form:

$$\sigma = Dw_\infty, \qquad (89)$$

where w_∞ is the root of (88). This yields simple expressions for phase velocity $\mathcal{C}_\infty$ and $\cos\gamma_\infty$:

$$\mathcal{C}_\infty = 1/|D\mathrm{Re}\,w_\infty|, \quad \cos\gamma_\infty = \epsilon(D/|D|)\mathrm{Im}\,w_\infty/\sqrt{(\mathrm{Im}\,w_\infty)^2 + D^{-2}}, \qquad (90)$$

where $\epsilon = \mathrm{Re}\,w_\infty/|\mathrm{Re}\,w_\infty| = \pm 1$.

Thus, phase velocity $\mathcal{C} \to 0$ for $|D| \to \infty$, for both *P* and *SV* waves. The behaviour of γ_∞ is more complex. Only for Im $w_\infty = 0$ (e.g. for isotropic viscoelastic medium) is $\gamma_\infty = 90°$. For anisotropic viscoelastic media, however, Im $w_\infty \neq 0$, and γ_∞ deviates from 90°. The deviation is positive for one sign of *D* and negative for the other sign of *D*.

3.5.5 Polarization of P and SV waves

It is not difficult to show from (38) that *P* and *SV* plane waves propagating in viscoelastic anisotropic or isotropic media are generally elliptically polarized. This is valid both for homogeneous and inhomogeneous plane waves.

4 CONCLUDING REMARKS

The importance of studies of wave propagation in viscoelastic anisotropic media in seismology and seismic exploration (for example in shallow consolidated sediments, reservoir rocks, zones of partial melting, mining areas, etc.) is indubitable. In this paper we have studied properties of homogeneous and inhomogeneous plane waves propagating in an unbounded viscoelastic anisotropic medium in an arbitrarily specified direction **N**. We used the mixed specification of the slowness vector discussed in Section 2.3. Although the mixed specification can be used for unrestricted anisotropy, we investigated special cases which can be treated analytically. Simple and transparent analytical expressions were derived and analysed in great detail. The analytical expressions are used together with solutions of algebraic equations of the sixth degree in the companion paper (Červený & Pšenčík 2005) to study numerically homogeneous and inhomogeneous plane waves in perfectly elastic or viscoelastic, isotropic or anisotropic media.

The proposed algorithms could be extended even to rheologically more complicated media (poroviscoelasticity, Biot, etc.). In most of these extensions, the degree of the algebraic equation in σ, analogous to (24), increases from six to eight.

It was shown that attenuation angle γ is a quantity which should be sought and not used as an *a priori* given parameter in the studies of *P*, *SV* and *SH* plane waves. The parameter γ, used as an input parameter, complicates computations considerably, and its use sometimes even leads to non-physical results. The fact that γ cannot be used quite freely is known already from studies of inhomogeneous plane waves propagating in perfectly elastic media, where γ can attain only a certain constant value. In the case of perfectly elastic anisotropic media the value of γ is generally different from 90°, in perfectly elastic isotropic media it is exactly 90°. Other values of γ are forbidden. On the other hand, the inhomogeneity parameter *D* introduced in the paper can be used universally and has all the necessary properties of a parameter of the problem.

Similarly, for *P*, *S1* and *S2* waves, propagating in an arbitrary direction (generally outside symmetry planes) in a viscoelastic medium of unrestricted anisotropy, unit vector **M** in the direction of attenuation vector **A** should be sought and not assumed to be known. Instead of **M**, the unit vector **m**, perpendicular to **n**, and inhomogeneity parameter *D*, should be specified (see eq. (12) and the text relevant to it).

We have only considered plane waves throughout the paper. Knowledge of plane wave algorithms, however, is a necessary prerequisite for the solution of more general problems of wave propagation (for example point sources, Green's functions) in inhomogeneous layered media (asymptotic high-frequency methods, ray methods).

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