EXACT ELASTODYNAMIC GREEN FUNCTIONS FOR SIMPLE TYPES OF ANISOTROPY DERIVED FROM HIGHER-ORDER RAY THEORY

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Summary: *Using higher-order ray theory, we derived exact elastodynamic Green functions for three simple types of homogeneous anisotropy. The first type displays an orthorhombic symmetry, the other two types display transverse isotropy. In all cases, the slowness surfaces of waves are either ellipsoids, spheroids or spheres. All three Green functions are expressed by a ray series with a finite number of terms. The Green functions can be written in explicit and elementary form similar to the Stokes solution for isotropy. In two Green functions, the higher-order ray approximations form a near-singularity term, which is significant near a kiss singularity. In the third Green function, the higher-order ray approximations also form a near-field term, which is significant near the point source. No effect connected with the line singularity was observed.*

1. INTRODUCTION

Vavryčuk and Yomogida (1995) showed that the elastodynamic Green function for isotropic, homogeneous and unbounded media can be calculated using higher-order ray theory and can be expressed in the form of a ray series. The ray series is finite and has only three non-zero terms including the zeroth-order term. The ray formula is exact and coincides with the closed-form solution found by Stokes (see *Aki and Richards, 1980, Eq. 4.23*; *Mura, 1993, Eq. 9.34*). In contrast to isotropy, the problem is much more complicated for anisotropy. In this case, the exact Green function is calculated by an integral over the slowness surface (*Buchwald, 1959*; *Lighthill, 1960*; *Burridge, 1967*; *Yeatts, 1984*; *Every and Kim, 1994*; *Wang and Achenbach, 1994*) and cannot be generally expressed in closed form. In using higher-order ray theory (*Červený et al., 1977*; *Vavryčuk, 1997*; *Vavryčuk, 1999b*; *Červený, 2000*), the Green function for general anisotropy is no longer expressed by a finite ray expansion. The higher-order ray theory may even fail to describe the Green function correctly, specifically in the directions of conical points or triplications of wavefronts. Nevertheless, it is worth finding at least some simple types of anisotropy for which the Green function can be described by the ray series. This is essential for understanding the applicability of the higher-order ray theory. Obviously, this is also important for a better understanding of the behaviour of waves in anisotropic media and for testing the accuracy of various approximate formulas for the Green functions in anisotropic media.

In this paper, we shall apply higher-order ray theory to calculating the exact Green functions for three special types of anisotropy. The Green functions for the first two types of anisotropy are already known being previously studied by *Payton (1983)* and *Burridge et al. (1993)* by using other

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methods. The form of the Green function for the third type of anisotropy has not been published yet. The anisotropy under study is very simple in all three cases. It is physically realisable, but still rather artificial with no experimental evidence to support its existence. Nevertheless, it is worth to study this anisotropy because it simplifies the problem while still preserving some essential features of wave propagation in anisotropic media. The first type displays an orthorhombic symmetry, the other two types display transverse isotropy. In all cases, the slowness surfaces of waves are either ellipsoids, spheroids or spheres. Polarization vectors of waves are also very simple: they have either a constant direction irrespective to the phase normal or they behave in the same way as in isotropy. Since the slowness surfaces have no parabolic points, we cannot study effects connected with a triplication of the wavefront. However, we can study effects connected with an intersection or contact of the slowness surfaces called singularities (*Crampin and Yedlin, 1981*; *Crampin, 1991*). The singularities are frequent in anisotropy but not known in isotropy. In particular, we shall study the effects connected with the line and kiss singularities (*Vavryčuk, 1999a*). We shall discuss the properties of the Green function for the mentioned types of anisotropic media and compare them with the Green function for isotropy.

2. HIGHER-ORDER RAY THEORY

The Green function for anisotropic, homogeneous, unbounded and elastic media satisfies the equation

$$\rho\ddot{G}_{in} - c_{ijkl}G_{kn,jl} = \delta_{in}\delta(\mathbf{x})\delta(t)\,, \tag{1}$$

where $G_{in} = G_{in}(\mathbf{x},t)$ is the symmetric Green tensor, ρ is the density, c_{ijkl} is the elasticity tensor, δ_{in} is the Kronecker delta, and $\delta(t)$ is the Dirac delta function. The Einstein summation convention is applied. The point source is located at the origin of coordinates.

Three waves propagate in anisotropic media. We denote them $W1$, $W2$ and $W3$. For each wave we seek a solution in the form of a ray series (*Červený et al., 1977, Eq. 5.2*)

$$G_{kl}(\mathbf{x},t) = \sum_{K=0}^{\infty} U_{kl}^{(K)}(\mathbf{x})\, f^{(K)}(t-\tau(\mathbf{x}))\ , \tag{2}$$

where K denotes the order of the ray approximation, $U_{kl}^{(K)}(\mathbf{x})$ is the ray amplitude tensor, $\tau(\mathbf{x})$ is the traveltime and $f^{(K)}(t)$ is the time function.

The zeroth-order term of the ray expansion reads (*Vavryčuk, 1997, Eqs 8 and 9*)

$$U_{kl}^{(0)}(\mathbf{x}) = \frac{1}{4\pi\rho}\frac{1}{\sqrt{K_p}\,v^2}\frac{g_k g_l}{\tau}\ , \qquad f^{(0)}(t) = \delta(t)\ . \tag{3}$$

Here K_p is the Gaussian curvature of the slowness surface, $\mathbf{g}$ is the unit polarization vector calculated as the eigenvector of the Christoffel tensor

$$\Gamma_{jk} = a_{ijkl}p_i p_l \tag{4}$$

and v is the group velocity. Components of the group velocity vector are expressed as

$$v_i = a_{ijkl} g_j g_k p_l \ , \tag{5}$$

where $a_{ijkl} = c_{ijkl} / \rho$ is the density-normalized elasticity tensor, $\mathbf{p} = \mathbf{p}(\mathbf{n})$ is the slowness vector, and $\mathbf{n}$ is the unit phase normal. In (3) we consider the positive Gaussian curvature, hence the slowness surfaces under study are always convex. For other shapes of the slowness surface, formula (3) should be modified (see *Burridge, 1967*; *Červený, 2000*).

To calculate the complete ray expansion of the Green function (2) we have to specify higher-order time functions and higher-order ray amplitudes. The higher-order time functions are expressed as follows

$$f^{(1)}(t) = H(t) \ , \qquad f^{(K)}(t) = \frac{t^{K-1}}{(K-1)!} H(t) \quad \text{for } K > 1 \ , \tag{6}$$

and the higher-order ray amplitudes can be calculated recursively by differentiating the lower-order ray amplitudes (*Vavryčuk, 1997, Eq. 7*):

$$U_{kn}^{W1(K)} = U_{kn}^{W1(K)\perp} + U_{kn}^{W1(K)\parallel} \ , \qquad K > 0 \ , \tag{7a}$$

$$U_{kn}^{W1(K)\perp} = \left[M_{in}\left(U_{kn}^{W1(K-1)}\right) - L_{in}\left(U_{kn}^{W1(K-2)}\right)\right] \left\{ \frac{g_i^{W2} g_k^{W2}}{G^{W2} - G^{W1}} + \frac{g_i^{W3} g_k^{W3}}{G^{W3} - G^{W1}} \right\} \ , \tag{7b}$$

$$U_{kn}^{W1(K)\parallel} = \frac{\tau^{W1}}{2K} \left[M_{in}\left(U_{kn}^{W1(K)\perp}\right) - L_{in}^{W1(K-1)} \right] g_i^{W1} g_k^{W1} \ , \tag{7c}$$

where differential operators $M_{jn}\left(U_{kn}^{(K)}\right)$ and $L_{jn}\left(U_{kn}^{(K)}\right)$ are defined as follows:

$$M_{jn}\left(U_{kn}^{(K)}\right) = a_{ijkl}\left[p_i U_{kn,l}^{(K)} + p_l U_{kn,i}^{(K)} + p_{i,l} U_{kn}^{(K)} \right] \ ,$$

$$L_{jn}\left(U_{kn}^{(K)}\right) = a_{ijkl} U_{kn,il}^{(K)} \ . \tag{8}$$

Quantities $U_{kn}^{W1(K)\perp}$ and $U_{kn}^{W1(K)\parallel}$ are called the additional and principal components of amplitude $U_{kn}^{W1(K)}$, and quantities $G^{W1} = 1$, G^{W2} and G^{W3} are the eigenvalues of the Christoffel tensor Γ_{jk}. Note that the formula (7c) was obtained by solving the transport equation for point sources in homogeneous media (see *Vavryčuk and Yomogida, 1996*).

For simple types of anisotropy, we can perform recursive differentiation (7) analytically, and thus we can obtain an explicit analytical ray expansion of the Green function. Since the calculation of higher-order ray amplitudes is rather extensive and tedious, we shall not present it in detail. In Appendix B we present some auxiliary formulas necessary for the calculation and in Sections 3-5 we present the final formulas.

3. ANISOTROPIC MEDIUM I (A-I)

A. Definition and basic quantities

Let us consider a special type of *orthorhombic medium* with the following density-normalised elastic parameters:

$$\mathbf{a} = \begin{bmatrix} a_{11} & -a_{66} & -a_{55} & 0 & 0 & 0 \\ & a_{22} & -a_{44} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix}, \tag{9}$$

where the two-index Voigt notation has been used. This anisotropy must satisfy the following stability conditions to be physically realisable (*Helbig, 1994, Eqs 5.3, 5.16 and 5.17*):

$$a_{11} \geq 0\,,\ a_{22} \geq 0\,,\ a_{33} \geq 0\,,\ a_{44} \geq 0\,,\ a_{55} \geq 0\,,\ a_{66} \geq 0\,,\ a_{11}a_{22} > a_{66}^2\,,$$

$$a_{11}a_{33} > a_{55}^2\,,\ a_{22}a_{33} > a_{44}^2\,,$$

$$a_{11}a_{22}a_{33} - 2a_{44}a_{55}a_{66} - a_{44}^2 a_{11} - a_{55}^2 a_{22} - a_{66}^2 a_{33} > 0\,. \tag{10}$$

The basic quantities for this medium are summarised as follows:

Christoffel tensor

$$\Gamma_{11} = a_{11}p_1^2 + a_{66}p_2^2 + a_{55}p_3^2\,,\ \Gamma_{22} = a_{66}p_1^2 + a_{22}p_2^2 + a_{44}p_3^2\,,$$

$$\Gamma_{33} = a_{55}p_1^2 + a_{44}p_2^2 + a_{33}p_3^2\,,\ \Gamma_{12} = 0\,,\ \Gamma_{13} = 0\,,\ \Gamma_{23} = 0\,. \tag{11}$$

Phase velocities

$$c_1 = \sqrt{a_{55}n_1^2 + a_{44}n_2^2 + a_{33}n_3^2}\,,\ c_2 = \sqrt{a_{11}n_1^2 + a_{66}n_2^2 + a_{55}n_3^2}\,,$$

$$c_3 = \sqrt{a_{66}n_1^2 + a_{22}n_2^2 + a_{44}n_3^2}\,, \tag{12}$$

where $\mathbf{n}$ is the unit phase normal.

A-I: slowness surfaces

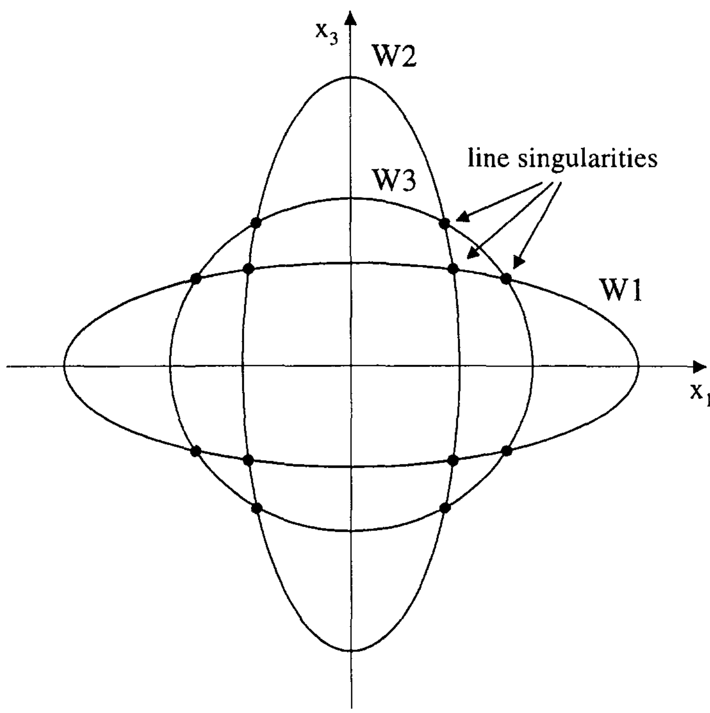

Fig. 1: The $x_1 - x_3$ section of slowness surfaces under anisotropy I. Parameters of the medium are: $a_{11} = 28$, $a_{22} = 23$, $a_{33} = 32$, $a_{44} = 12$, $a_{55} = 4$, $a_{66} = 10$. Dots show the line singularities.

Group velocities

$$v_1 = \sqrt{\frac{a_{55}^2 n_1^2 + a_{44}^2 n_2^2 + a_{33}^2 n_3^2}{a_{55} n_1^2 + a_{44} n_2^2 + a_{33} n_3^2}}\,, \quad v_2 = \sqrt{\frac{a_{11}^2 n_1^2 + a_{66}^2 n_2^2 + a_{55}^2 n_3^2}{a_{11} n_1^2 + a_{66} n_2^2 + a_{55} n_3^2}}\,,$$

$$v_3 = \sqrt{\frac{a_{66}^2 n_1^2 + a_{22}^2 n_2^2 + a_{44}^2 n_3^2}{a_{66} n_1^2 + a_{22} n_2^2 + a_{44} n_3^2}}\,. \tag{13}$$

Polarization vectors

$$\mathbf{g}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{g}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{14}$$

Gaussian curvatures of the slowness surface

$$K_1 = \frac{a_{33}a_{44}a_{55}}{v_1^4}\ , \quad K_2 = \frac{a_{11}a_{55}a_{66}}{v_2^4}\ , \quad K_3 = \frac{a_{22}a_{44}a_{66}}{v_3^4}\ . \tag{15}$$

The slowness surfaces of all three waves are ellipsoids (see Fig. 1). The polarization of waves is exceptionally simple: polarization vectors are constant irrespective of the position of the observation point. The slowness surfaces can intersect along curves, which form a line (or intersection) singularity (*Crampin and Yedlin, 1981*).

B. Green function and its properties

The exact analytical formula for the Green function can be expressed as follows:

$$G_{kl}(\mathbf{x},t) = \frac{1}{4\pi\rho}\left\{\frac{1}{\sqrt{a_{33}a_{44}a_{55}}}\frac{\delta_{k3}\delta_{l3}}{\tau_1}\delta(t-\tau_1) + \frac{1}{\sqrt{a_{11}a_{55}a_{66}}}\frac{\delta_{k1}\delta_{l1}}{\tau_2}\delta(t-\tau_2) + \right.$$
$$\left. + \frac{1}{\sqrt{a_{22}a_{44}a_{66}}}\frac{\delta_{k2}\delta_{l2}}{\tau_3}\delta(t-\tau_3)\right\}\ , \tag{16}$$

where

$$\tau_1 = r\sqrt{\frac{N_1^2}{a_{55}} + \frac{N_2^2}{a_{44}} + \frac{N_3^2}{a_{33}}}\ , \quad \tau_2 = r\sqrt{\frac{N_1^2}{a_{11}} + \frac{N_2^2}{a_{66}} + \frac{N_3^2}{a_{55}}}\ , \quad \tau_3 = r\sqrt{\frac{N_1^2}{a_{66}} + \frac{N_2^2}{a_{22}} + \frac{N_3^2}{a_{44}}}$$

are the traveltimes of the three waves, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance from the source to the receiver, δ_{kl} is the Kronecker delta, and $N_k = \frac{x_k}{r}$ is the unit direction vector to the receiver.

The Green function (16) is extremely simple. It only consists of the zeroth-order terms of the *W*1-, *W*2- and *W*3-ray series. All higher-order terms of the *W*1-, *W*2- and *W*3-ray expansions are zero. The zeroth-order terms correspond physically to far-field waves. Their time function is the Dirac delta function, hence their amplitude is non-zero only at the times of their arrival. The amplitude depends on distance r from the source as $1/r$. Interestingly, no near-field terms known from isotropy are present in the wavefield. Also no effects connected with the intersection singularity are observed. The reason is that the polarization vectors of the waves behave regularly in the singularity as well as at the point source. The Green function is probably the simplest Green function, which can be found for anisotropic media.

4. ANISOTROPIC MEDIUM II (A-II)

A. Definition and basic quantities

Let us consider a *transversely isotropic* medium with a vertical axis of symmetry and the following density-normalised elastic parameters expressed in two-index notation:

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{11}-2a_{66} & -a_{44} & 0 & 0 & 0 \\ & a_{11} & -a_{44} & 0 & 0 & 0 \\ & & a_{33} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{44} & 0 \\ & & & & & a_{66} \end{bmatrix} . \tag{17}$$

This medium represents a special type of transverse isotropy, obtained from general transverse isotropy by substituting parameter a_{13} by value $-a_{44}$. This transverse isotropy must satisfy the following stability conditions to be physically realisable (*Backus, 1962, Eq. 20*):

$$a_{33} \geq 0 \, , \; a_{44} \geq 0 \, , \; a_{66} \geq 0 \, , \; a_{11}-a_{66} \geq 0 \quad \text{and} \quad a_{33}(a_{11}-a_{66}) \geq a_{44}^2 \, . \tag{18}$$

We summarise the basic wave quantities as follows:

Christoffel tensor

$$\Gamma_{11} = a_{11}p_1^2 + a_{66}p_2^2 + a_{44}p_3^2 \, , \quad \Gamma_{22} = a_{66}p_1^2 + a_{11}p_2^2 + a_{44}p_3^2 \, ,$$

$$\Gamma_{33} = a_{44}\left(p_1^2 + p_2^2\right) + a_{33}p_3^2 \, , \quad \Gamma_{12} = (a_{11}-a_{66})\, p_1 p_2 \, , \quad \Gamma_{13} = 0 \, , \quad \Gamma_{23} = 0 \, . \tag{19}$$

Phase velocities

$$c_1 = \sqrt{a_{44}\left(n_1^2 + n_2^2\right) + a_{33}n_3^2} \, , \quad c_2 = \sqrt{a_{11}\left(n_1^2 + n_2^2\right) + a_{44}n_3^2} \, ,$$

$$c_3 = \sqrt{a_{66}\left(n_1^2 + n_2^2\right) + a_{44}n_3^2} \, . \tag{20}$$

Group velocities

$$v_1 = \sqrt{\frac{a_{44}^2\left(n_1^2 + n_2^2\right) + a_{33}^2 n_3^2}{a_{44}\left(n_1^2 + n_2^2\right) + a_{33}n_3^2}} \, , \quad v_2 = \sqrt{\frac{a_{11}^2\left(n_1^2 + n_2^2\right) + a_{44}^2 n_3^2}{a_{11}\left(n_1^2 + n_2^2\right) + a_{44}n_3^2}} \, ,$$

$$v_3 = \sqrt{\frac{a_{66}^2\left(n_1^2 + n_2^2\right) + a_{44}^2 n_3^2}{a_{66}\left(n_1^2 + n_2^2\right) + a_{44}n_3^2}} \, . \tag{21}$$

Polarization vectors

$$\mathbf{g}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{g}_2 = \frac{1}{\sqrt{N_1^2 + N_2^2}} \begin{bmatrix} N_1 \\ N_2 \\ 0 \end{bmatrix}, \quad \mathbf{g}_3 = \frac{1}{\sqrt{N_1^2 + N_2^2}} \begin{bmatrix} N_2 \\ -N_1 \\ 0 \end{bmatrix}. \tag{22}$$

Gaussian curvatures of the slowness surface

$$K_1 = \frac{a_{44}^2 a_{33}}{v_1^4}, \quad K_2 = \frac{a_{11}^2 a_{44}}{v_2^4}, \quad K_3 = \frac{a_{66}^2 a_{44}}{v_3^4}. \tag{23}$$

A-II: slowness surfaces

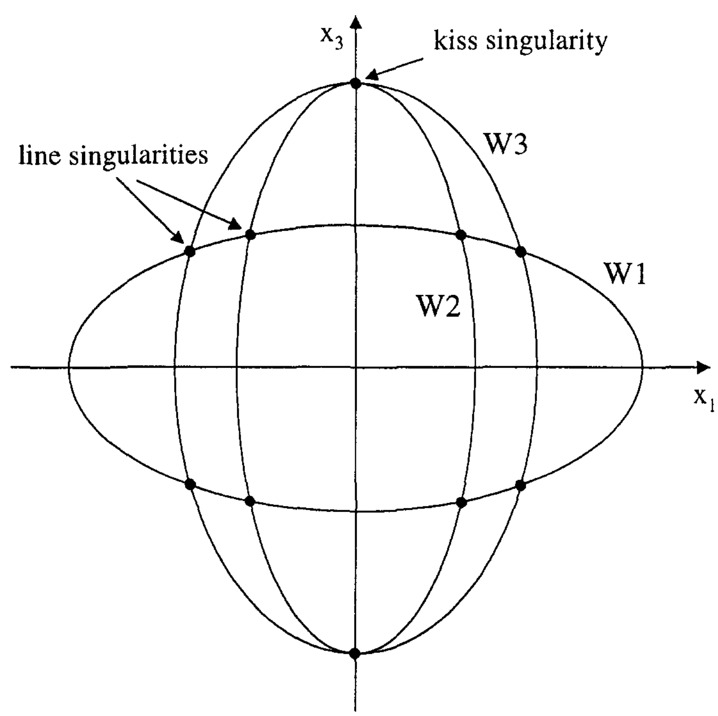

Fig. 2: The $x_1 - x_3$ section of slowness surfaces under anisotropy II. Parameters of the medium are: $a_{11} = 23$, $a_{33} = 16$, $a_{44} = 4$, $a_{66} = 10$. Dots show the singularities.

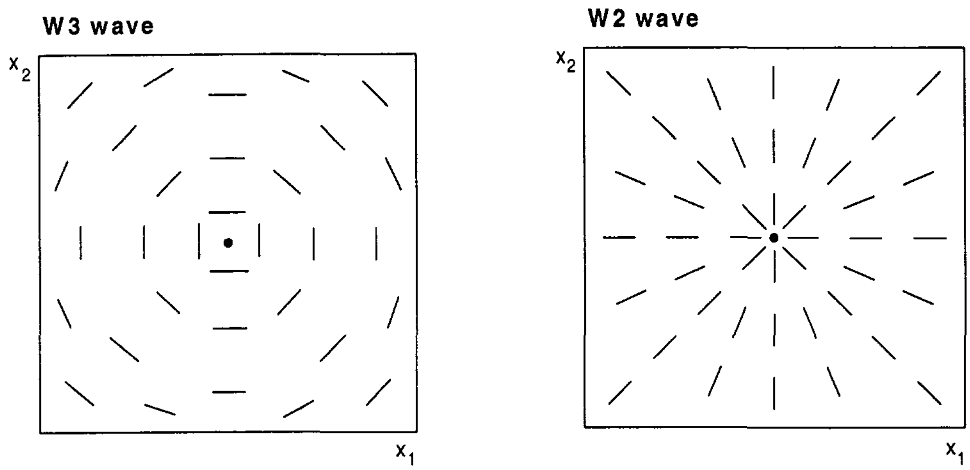

Fig. 3: Polarization vectors near a kiss singularity in the $x_1 - x_2$ plane.

Slowness surfaces of all three waves are spheroids with axis of rotation along the z-axis (see Fig. 2). For $a_{66} = a_{44}$, the slowness surface of the $W3$ wave becomes a sphere. The $W1$ wave has a constant polarization vector parallel to the symmetry axis. The polarization vectors of the $W2$ and $W3$ waves are always perpendicular to the symmetry axis. The slowness surfaces of the $W1$ and $W2$ waves intersect each other along two circles. The slowness surfaces of the $W2$ and $W3$ waves touch tangentially in the symmetry axis direction, and thus form a kiss singularity (*Crampin and Yedlin, 1981*; *Vavryčuk, 1999a*). The behaviour of the $W2$ and $W3$ polarization vectors is quite anomalous in the vicinity of this singularity (see Fig. 3). For the strictly singular direction, the polarization vectors of the $W2$ and $W3$ waves are not defined.

B. Green function and its properties

The exact analytical formula for the Green function can be expressed as follows:

$$G_{kl}(\mathbf{x},t) = \frac{1}{4\pi\rho}\left\{\frac{1}{a_{44}\sqrt{a_{33}}}\frac{g_{1k}g_{1l}}{\tau_1}\delta(t-\tau_1) + \frac{1}{a_{11}\sqrt{a_{44}}}\frac{g_{2k}g_{2l}}{\tau_2}\delta(t-\tau_2) + \right.$$
$$\left. + \frac{1}{a_{66}\sqrt{a_{44}}}\frac{g_{3k}g_{3l}}{\tau_3}\delta(t-\tau_3) + \frac{1}{\sqrt{a_{44}}}\frac{g_{2k}g_{2l} - g_{3k}g_{3l}}{R^2}\int_{\tau_2}^{\tau_3}\delta(t-\tau)d\tau\right\}, \quad (24)$$

where

$$\tau_1 = \frac{r}{\sqrt{a_{44}}}\sqrt{N_1^2 + N_2^2 + \frac{a_{44}}{a_{33}}N_3^2}\ , \quad \tau_2 = \frac{r}{\sqrt{a_{11}}}\sqrt{N_1^2 + N_2^2 + \frac{a_{11}}{a_{44}}N_3^2}\ ,$$

$$\tau_3 = \frac{r}{\sqrt{a_{66}}}\sqrt{N_1^2 + N_2^2 + \frac{a_{66}}{a_{44}}N_3^2} \ ,$$

are traveltimes of the $W1$, $W2$ and $W3$ waves, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance from the source to the receiver, $R = \sqrt{x_1^2 + x_2^2}$ is the distance of the receiver from the symmetry axis of the medium, and $N_k = \frac{x_k}{r}$ is the unit direction vector to the receiver.

The Green function (24) consists of the zeroth-order term of the $W1$-ray expansion and of the zeroth- and first-order terms of the $W2$- and $W3$-ray expansions. The zeroth-order terms physically mean the far-field waves, the first-order terms of the $W1$ and $W2$ expansions couple into the fourth term in (24) called the near-singularity term (see *Vavryčuk, 1999a*). The time dependence of the far-field waves is the Dirac delta function. The near-singularity term couples the $W2$ and $W3$ waves, being non-zero between the arrivals of these waves (see Fig. 4). Interestingly, no coupling between $W1$ and $W2$, or between $W1$ and $W3$ waves is present. Also, no near-field term is observed in this medium. The reason for the absence of the near-field term lies in the exceptionally simple polarization of the $W1$ wave. Since the $W1$-polarization vector is constant irrespective of the position of the observation point, it displays no singular behaviour at the source. The near-singularity term is caused by the existence of the kiss singularity in the symmetry axis direction of transverse isotropy. Near this direction, the behaviour of the polarization vectors is anomalous (see Fig. 3). The amplitude of the near-singularity term decreases with distance R from the singularity as $1/R^2$. Therefore, the near-singularity term is significant in the vicinity of the singularity but negligible far from the singularity. In the singularity direction ($R = 0$), the amplitude of this term diverges and the waveform of the near-singularity term becomes the Dirac delta function, similarly as the far-field waves. Also the amplitude decrease becomes the same as for the far-field waves (see *Vavryčuk, 1999a*).

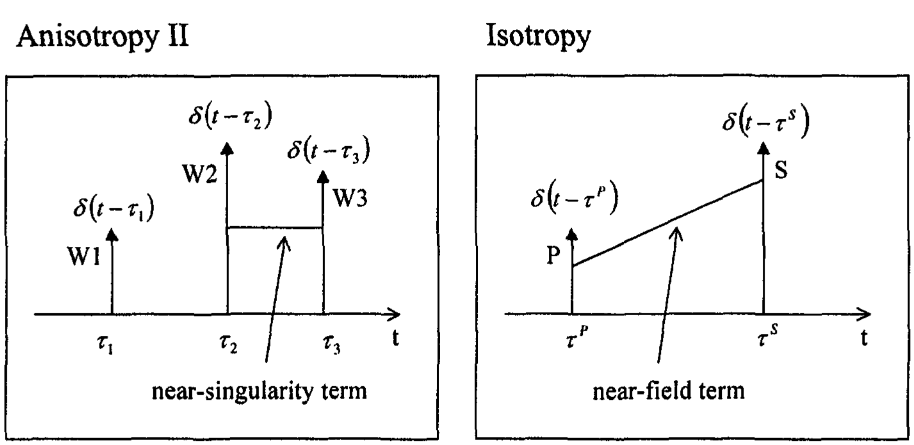

Fig. 4: Comparison of waveforms of the Green function for anisotropy II and isotropy.

5. ANISOTROPIC MEDIUM III (A-III)

A. Definition and basic quantities

Let us consider a *transversely isotropic* medium with a vertical axis of symmetry and the following density-normalised elastic parameters:

$$\mathbf{a}=\begin{bmatrix} a_{11} & a_{11}-2a_{66} & a_{11}-2a_{44} & 0 & 0 & 0 \\ & a_{11} & a_{11}-2a_{44} & 0 & 0 & 0 \\ & & a_{11} & 0 & 0 & 0 \\ & & & a_{44} & 0 & 0 \\ & & & & a_{44} & 0 \\ & & & & & a_{66} \end{bmatrix} . \tag{25}$$

This transverse isotropy is very similar to isotropy. For $a_{66}=a_{44}$, the medium becomes strictly isotropic. The stability conditions for this medium are expressed as follows (*Backus, 1962, Eq. 20*):

$$a_{44}\geq 0\,,\ a_{66}\geq 0\,,\ a_{11}-a_{66}\geq 0 \ \text{and}\ a_{11}(a_{11}-a_{66})\geq (a_{11}-2a_{44})^2 \,. \tag{26}$$

We summarise the basic wave quantities as follows:

Christoffel tensor

$$\Gamma_{11}=a_{11}p_1^2+a_{66}p_2^2+a_{44}p_3^2\,,\quad \Gamma_{22}=a_{66}p_1^2+a_{11}p_2^2+a_{44}p_3^2\,,$$

$$\Gamma_{33}=a_{44}\left(p_1^2+p_2^2\right)+a_{11}p_3^2\,,\quad \Gamma_{12}=(a_{11}-a_{66})\,p_1p_2\,,$$

$$\Gamma_{13}=(a_{11}-a_{44})\,p_1p_3\,,\ \Gamma_{23}=(a_{11}-a_{44})\,p_2p_3\,. \tag{27}$$

Phase velocities

$$c_1=\sqrt{a_{11}}\,,\quad c_2=\sqrt{a_{44}}\,,\quad c_3=\sqrt{a_{66}\left(n_1^2+n_2^2\right)+a_{44}n_3^2}\,. \tag{28}$$

Group velocities

$$v_1=\sqrt{a_{11}}\,,\quad v_2=\sqrt{a_{44}}\,,\quad v_3=\sqrt{\frac{a_{66}^2\left(n_1^2+n_2^2\right)+a_{44}^2n_3^2}{a_{66}\left(n_1^2+n_2^2\right)+a_{44}n_3^2}}\,. \tag{29}$$

Polarization vectors

$$\mathbf{g}_1=\begin{bmatrix}N_1\\N_2\\N_3\end{bmatrix},\quad \mathbf{g}_2=\frac{-1}{\sqrt{N_1^2+N_2^2}}\begin{bmatrix}-N_1N_3\\-N_2N_3\\N_1^2+N_2^2\end{bmatrix},\quad \mathbf{g}_3=\frac{1}{\sqrt{N_1^2+N_2^2}}\begin{bmatrix}N_2\\-N_1\\0\end{bmatrix}. \tag{30}$$

Gaussian curvatures of the slowness surface

$$K_1 = a_{11} \ , \quad K_2 = a_{44} \ , \quad K_3 = \frac{a_{66}^2 a_{44}}{v_3^4} \ . \tag{31}$$

The slowness surfaces of the *W*1 and *W*2 waves are spheres (see Fig. 5), the slowness surface of the *W*3 wave is a spheroid whose rotation axis is along the *z*-axis. The polarization vectors of the *W*1, *W*2 and *W*3 waves are identical to those of the *P*, *SV* and *SH* waves under isotropy. The *W*1 wave is longitudinal, the *W*2 and *W*3 waves are transverse. The *W*2 wave is polarized in the plane defined by the symmetry axis and a ray. The *W*3-wave polarization vector is always perpendicular to the symmetry axis of the medium. The slowness surfaces of the *W*1 and *W*2 waves or *W*1 and *W3* waves cannot intersect. Similarly to the A-II medium, the slowness surfaces of the *W*2 and *W*3 waves touch in the symmetry axis direction forming a kiss singularity. The behaviour of the *W*2 and *W*3 polarization vectors is again anomalous in the vicinity of the kiss singularity. Moreover, the polarization of the *W*1 wave is anomalous in the vicinity of the source, displaying a similar pattern as the *W*2 wave near a kiss singularity (see Fig. 3).

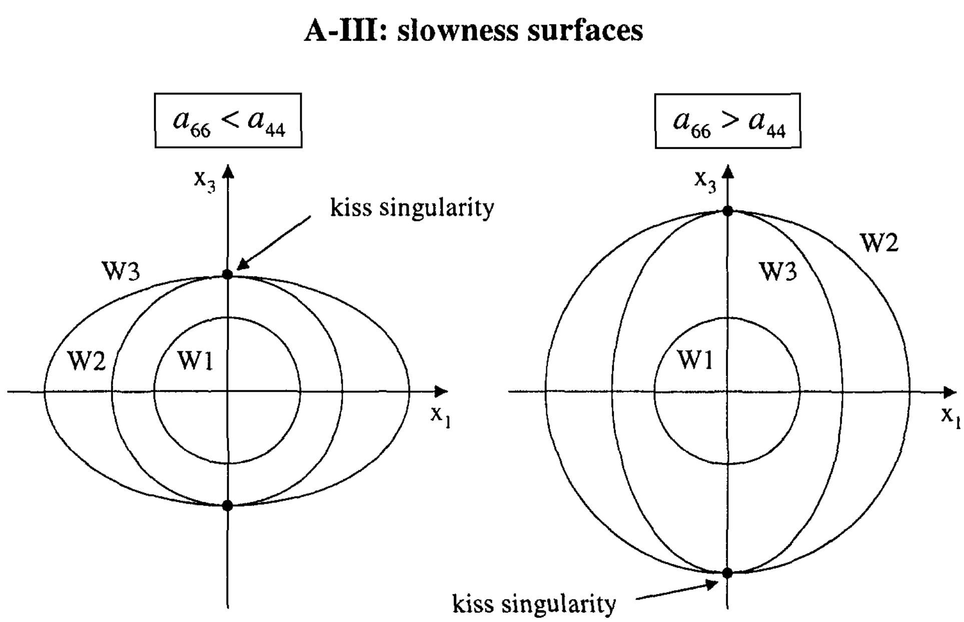

Fig. 5: The $x_1 - x_3$ section of slowness surfaces under anisotropy III. Parameters of the medium are: $a_{11} = 25$, $a_{44} = 10$, $a_{66} = 4$ (left-hand plot), and $a_{11} = 25$, $a_{44} = 4$, $a_{66} = 10$ (right-hand plot). Dots show the kiss singularities.

B. Green function and its properties

The exact analytical formula for the Green function can be expressed as follows:

$$G_{kl}(\mathbf{x},t) = \frac{1}{4\pi\rho}\left\{ \frac{1}{\sqrt{a_{11}^3}} \frac{g_{1k}g_{1l}}{\tau_1}\delta(t-\tau_1) + \frac{1}{\sqrt{a_{44}^3}} \frac{g_{2k}g_{2l}}{\tau_2}\delta(t-\tau_2) + \right.$$

$$+ \frac{1}{a_{66}\sqrt{a_{44}}} \frac{g_{3k}g_{3l}}{\tau_3}\delta(t-\tau_3) + \frac{1}{\sqrt{a_{44}}} \frac{g_{3k}^{\perp}g_{3l}^{\perp} - g_{3k}g_{3l}}{R^2} \times$$

$$\left. \times \int_{\tau_2}^{\tau_3} \delta(t-\tau)d\tau + \frac{3g_{1k}g_{1l} - \delta_{kl}}{r^3} \int_{\tau_1}^{\tau_2} \tau\delta(t-\tau)d\tau \right\} , \qquad (32)$$

where

$$\mathbf{g}_3^{\perp} = \frac{1}{\sqrt{N_1^2 + N_2^2}} \begin{bmatrix} N_1 \\ N_2 \\ 0 \end{bmatrix} , \quad \tau_1 = \frac{r}{\sqrt{a_{11}}} , \quad \tau_2 = \frac{r}{\sqrt{a_{44}}} ,$$

$$\tau_3 = \frac{r}{\sqrt{a_{66}}} \sqrt{N_1^2 + N_2^3 + \frac{a_{66}}{a_{44}} N_3^2} .$$

Here τ_1, τ_2 and τ_3 are the traveltimes of the $W1$, $W2$ and $W3$ waves, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the distance from the source to the receiver, $R = \sqrt{x_1^2 + x_2^2}$ is the distance of the receiver from the vertical axis, and $N_k = \frac{x_k}{r}$ is the unit direction vector to the receiver.

The Green function for the A-III medium (32) is more complicated than for the A-I or A-II media. It consists of the zeroth- and first-order terms of the $W1$-, $W2$- and $W3$-ray expansions and of the second-order terms of the $W1$- and $W2$-ray expansions. The zeroth-order terms represent the far-field waves (the first three terms in 32), the higher-order terms yield the near-singularity term (the fourth term in 32) and the near-field term (the fifth term in 32). The time dependence of the far-field waves is the Dirac delta function. The amplitude of the near-singularity term is non-zero between the arrivals of the $W2$ and $W3$ waves, and the amplitude of the near-field term is non-zero between the $W1$ and $W2$ waves (see Fig. 6). The near-singularity term has a form identical to that in the A-II medium. The near-field term has the same form as for isotropy (see Fig. 6). The amplitude of the near-singularity term decreases with distance R from the singularity as $1/R^2$. The amplitude of the near-field term decreases with distance r from the source as $1/r^2$.

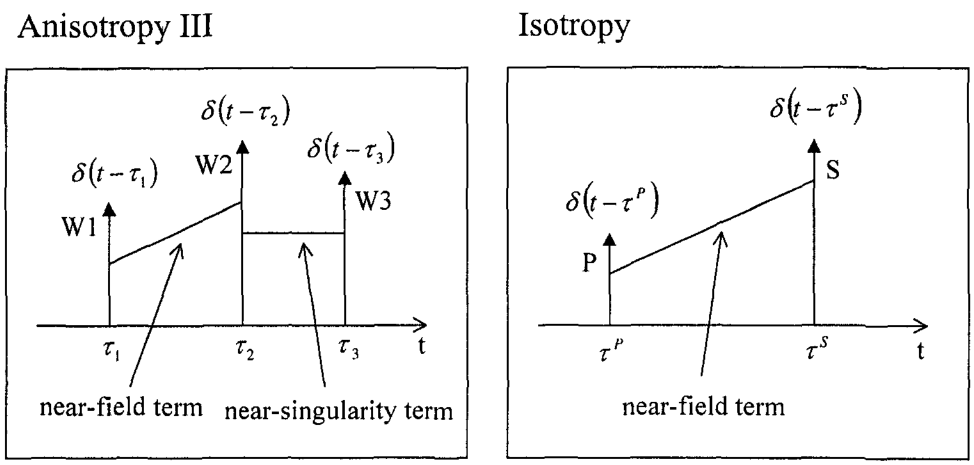

Fig. 6. Comparison of waveforms of the Green function for anisotropy III and isotropy.

6. CONCLUSIONS

Using higher-order ray theory we have calculated the exact Green functions for three simple types of anisotropy. The first two Green functions were originally derived by *Payton (1983)* and *Burridge et al. (1993)* by other methods. The third Green function has not been presented yet. Correctness of the Green functions was verified by inserting them into the elastodynamic equation. All three Green functions are expressed by a ray series with a finite number of terms. The presented Green functions are probably the simplest Green functions, which can be found for anisotropic media. They can be expressed in explicit and elementary form similar to the Stokes solution for isotropic media. For the first two types of anisotropy, the structure of the Green function is even simpler than that for isotropy.

The first medium (A-I) is strongly anisotropic displaying an orthorhombic symmetry. The slowness surfaces of all three waves propagating in the medium are ellipsoids. The polarization vectors of the waves are constant irrespective of the position of the observation point. The Green function is expressed only by the zeroth-order term of the ray series. All higher-order terms are zero. No near-field wave known from the isotropic Green function is observed. Also no coupling due to singularities is observed.

The second medium (A-II) is also strongly anisotropic displaying transverse isotropy. The slowness surfaces of all three waves are spheroids. The Green function is expressed by two non-zero terms of the ray series including the zeroth-order term. The Green function contains no near-field term, but does contain the *near-singularity term*, which is not present in the isotropic Green function. The near-singularity term is caused by the presence of the kiss singularity, which occurs in the symmetry axis direction of transverse isotropy. In this direction, the slowness surfaces of two slower waves touch and polarization of the waves is anomalous near this direction. In the A-II medium, also a line singularity can exist, but no additional term connected to this singularity appears in the Green function.

The third medium (A-III) displays transverse isotropy, which can be strong but also weak. The slowness surfaces are spheres for two waves and a spheroid for the third wave. The Green function consists of three non-zero terms of the ray series, similarly to the Green function for isotropy. The Green function contains the near-singularity term but also the near-field term. The near-singularity term has a form identical with that for the A-II medium. The near-field term has the same form as for isotropy. If the anisotropy vanishes, the Green function for A-III smoothly converges to the Green function for isotropy.

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APPENDIX A: ELASTOSTATIC GREEN FUNCTIONS

From the elastodynamic Green functions (16), (24) and (32) we can derive the elastostatic Green functions as follows:

$$G_{kl}^{S}(x) = \int_{-\infty}^{\infty} G_{kl}(x,t)\,dt \ .$$

For the A-I medium

$$G_{kl}^{S}(\mathbf{x}) = \frac{1}{4\pi\rho}\left\{\frac{1}{\sqrt{a_{33}a_{44}a_{55}}}\frac{\delta_{k3}\delta_{l3}}{\tau_1} + \frac{1}{\sqrt{a_{11}a_{55}a_{66}}}\frac{\delta_{k1}\delta_{l1}}{\tau_2} + \frac{1}{\sqrt{a_{22}a_{44}a_{66}}}\frac{\delta_{k2}\delta_{l2}}{\tau_3}\right\}, \quad \text{(A1)}$$

For the A-II medium

$$G_{kl}^{S}(\mathbf{x}) = \frac{1}{4\pi\rho}\left\{\frac{1}{\sqrt{a_{33}a_{44}}}\frac{\delta_{k3}\delta_{l3}}{R_1} + \frac{1}{\sqrt{a_{11}a_{44}}}\frac{1}{R_2\left(1-N_3^2\right)}\left[A_2(\delta_{kl}-\delta_{k3}\delta_{l3}) - B_2 g_{2k} g_{2l}\right] + \right.$$

$$\left. + \frac{1}{\sqrt{a_{66}a_{44}}}\frac{1}{R_3\left(1-N_3^2\right)}\left[A_3(\delta_{kl}-\delta_{k3}\delta_{l3}) - B_3 g_{3k} g_{3l}\right]\right\}, \quad \text{(A2)}$$

For the A-III medium

$$G_{kl}^{S}(\mathbf{x}) = \frac{1}{8\pi\rho}\left\{\frac{1}{a_{11}r}(\delta_{kl} - N_k N_l) + \frac{1}{a_{44}r}\left[\delta_{kl} + N_k N_l + \frac{4N_3^2}{\left(1-N_3^2\right)^2}(\delta_{kl}-\delta_{k3}\delta_{l3}) - \right.\right.$$

$$-2\frac{1+N_3^2}{\left(1-N_3^2\right)^2}\left(N_kN_l+N_3^2\delta_{kl}-N_3N_l\delta_{k3}-N_3N_k\delta_{l3}\right)\Bigg]+$$

$$+\frac{1}{\sqrt{a_{44}a_{66}}}\frac{2}{R_3\left(1-N_3^2\right)}[A_3(\delta_{kl}-\delta_{k3}\delta_{l3})-B_3g_{3k}g_{3l}]\Bigg\}\ , \tag{A3}$$

where

$$\tau_1=r\sqrt{\frac{N_1^2}{a_{55}}+\frac{N_2^2}{a_{44}}+\frac{N_3^2}{a_{33}}}\ ,\ \tau_2=r\sqrt{\frac{N_1^2}{a_{11}}+\frac{N_2^2}{a_{66}}+\frac{N_3^2}{a_{55}}}\ ,\ \tau_3=r\sqrt{\frac{N_1^2}{a_{66}}+\frac{N_2^2}{a_{22}}+\frac{N_3^2}{a_{44}}}\ ,$$

$$R_1=r\sqrt{N_1^2+N_2^2+\frac{a_{44}}{a_{33}}N_3^2}\ ,\ R_2=r\sqrt{N_1^2+N_2^2+\frac{a_{11}}{a_{44}}N_3^2}\ ,\ R_3=r\sqrt{N_1^2+N_2^2+\frac{a_{66}}{a_{44}}N_3^2}\ ,$$

$$A_2=N_1^2+N_2^2+\frac{a_{11}}{a_{44}}N_3^2\ ,\ A_3=N_1^2+N_2^2+\frac{a_{66}}{a_{44}}N_3^2\ ,$$

$$B_2=N_1^2+N_2^2+2\frac{a_{11}}{a_{44}}N_3^2\ ,\ B_3=N_1^2+N_2^2+2\frac{a_{66}}{a_{44}}N_3^2\ ,$$

$$\mathbf{g}_2=\frac{1}{\sqrt{N_1^2+N_2^2}}\begin{bmatrix}N_1\\N_2\\0\end{bmatrix},\ \mathbf{g}_3=\frac{1}{\sqrt{N_1^2+N_2^2}}\begin{bmatrix}N_2\\-N_1\\0\end{bmatrix},\ r=\sqrt{x_1^2+x_2^2+x_3^2}\ ,\ N_k=\frac{x_k}{r}\ .$$

Correctness of formulas (A1)-(A3) has been verified by inserting these formulas into the elastostatic equation.

APPENDIX B: AUXILIARY FORMULAS

For calculation of higher-order ray approximations we need the following relations and derivatives:

$$\mathbf{n}=\frac{1}{\sqrt{\frac{N_1^2}{A_W^2}+\frac{N_2^2}{B_W^2}+\frac{N_3^2}{C_W^2}}}\left(\frac{N_1}{A_W},\ \ \frac{N_2}{B_W},\ \ \frac{N_3}{C_W}\right)^T\ , \tag{B-1}$$

$$\mathbf{N}=\frac{1}{\sqrt{A_W^2n_1^2+B_W^2n_2^2+C_W^2n_3^2}}(A_Wn_1,\ \ B_Wn_2,\ \ C_Wn_3)^T\ , \tag{B-2}$$

$$\mathbf{p}^W = \frac{1}{c_W}(n_1, \quad n_2, \quad n_3)^T = \frac{1}{v_W}\left(\frac{N_1}{A_W}, \quad \frac{N_2}{B_W}, \quad \frac{N_3}{C_W}\right)^T , \tag{B-3}$$

$$\frac{\partial \tau}{\partial x_i} = p_i \ , \quad \frac{\partial r}{\partial x_i} = N_i \ , \quad \frac{\partial N_j}{\partial x_i} = \frac{\delta_{ij} - N_i N_j}{r} \ , \tag{B-4}$$

$$\left(\frac{\partial n_j}{\partial x_1}, \quad \frac{\partial n_j}{\partial x_2}, \quad \frac{\partial n_j}{\partial x_3}\right) = \frac{c_W}{\tau_W}\left(\frac{\delta_{1j} - n_1 n_j}{A_W}, \quad \frac{\delta_{2j} - n_2 n_j}{B_W}, \quad \frac{\delta_{3j} - n_3 n_j}{C_W}\right) , \tag{B-5}$$

where $W = 1,2,3$ means the type of the wave, $\mathbf{n}$ is the phase normal, $\mathbf{N}$ is the unit direction vector to a receiver, c and v are the phase and group velocities, $\mathbf{p}$ is the slowness vector, r is the distance between the source and receiver, and τ is the traveltime. Constants A_W, B_W and C_W are defined as follows:

For A-I

$$A_1 = a_{55} \, , \; B_1 = a_{44} \, , \; C_1 = a_{33} \ ,$$

$$A_2 = a_{11} \, , \; B_2 = a_{66} \, , \; C_2 = a_{55} \ ,$$

$$A_3 = a_{66} \, , \; B_3 = a_{22} \, , \; C_3 = a_{44} \ , \tag{B-6}$$

For A-II

$$A_1 = a_{44} \, , \; B_1 = a_{44} \, , \; C_1 = a_{33} \ ,$$

$$A_2 = a_{11} \, , \; B_2 = a_{11} \, , \; C_2 = a_{44} \ ,$$

$$A_3 = a_{66} \, , \; B_3 = a_{66} \, , \; C_3 = a_{44} \ , \tag{B-7}$$

For A-III

$$A_1 = a_{11} \, , \; B_1 = a_{11} \, , \; C_1 = a_{11} \ ,$$

$$A_2 = a_{44} \, , \; B_2 = a_{44} \, , \; C_2 = a_{44} \ ,$$

$$A_3 = a_{66} \, , \; B_3 = a_{66} \, , \; C_3 = a_{44} \ . \tag{B-8}$$

When calculating higher-order ray approximations for a specified type of wave we have to keep in mind that all calculations must be performed for a fixed phase normal, but not for a fixed ray direction.

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