PROPERTIES OF WEAK CONTRAST PP REFLECTION/TRANSMISSION COEFFICIENTS FOR WEAKLY ANISOTROPIC ELASTIC MEDIA

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Summary: Approximate formulae for PP reflection/transmission (R/T) displacement coefficients for weak contrast interfaces separating weakly anisotropic media of an arbitrary symmetry are presented. The coefficients have a form of a sum of a well-known approximate PP reflection/transmission coefficient for a weak contrast interface separating two background isotropic halfspaces and a correction due to weak anisotropy. The correction is controlled, linearly, by the so-called weak anisotropy (WA) parameters. The coefficients are defined with respect to an arbitrary isotropic background. The formulae are convenient for description of coefficients of reflection and transmission in low symmetry weakly anisotropic media as well as in media with higher symmetry (orthorhombic, hexagonal) with arbitrarily oriented axes of symmetry. For anisotropies of higher symmetry, for which approximate formulae of other authors exist, we discuss the differences between these and our formulae and effects of these differences on the accuracy of approximate coefficients. Performance of the approximate formulae for Rpp coefficients is tested on models with weak anisotropy and weak contrast interfaces as well as on models whose anisotropy and velocity contrast are in no way weak. Even in the latter case, the formulae yield satisfactory results. Accuracy of the approximate Rpp coefficient depending on the choice of the background is then investigated. It is shown that the Rpp coefficient can be described, approximately, by a formula, whose correction term due to anisotropy is independent of the S-wave background velocity and depends only slightly on the choice of the P-wave background velocity. The formulae for the Rpp coefficients commonly used in literature for higher symmetry anisotropies are obtained by linearization of the above mentioned formula. Linearization leads to the dependence of the Rpp coefficient on the choice of the S-wave background velocity and, generally, to a slight decrease of accuracy. Presented formulae are convenient for solving an inverse problem: determination of contrasts of parameters of anisotropic media surrounding an interface. Sensitivity of the approximate Rpp coefficient to basic weak anisotropy parameters is presented.

1. INTRODUCTION

Study of relation of the reflection (R) or, possibly, transmission (T) coefficients to parameters of the media surrounding the reflector is a basic part of the amplitude-versus-offset (AVO) or amplitude-versus-azimuth (AVA) analysis. In the case of a reflector surrounded by anisotropic media, formulae for the R/T coefficients are rather complicated if their analytic form exists at all.

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They can be simplified substantially if the anisotropy and the velocity contrast across the reflector are weak. Since this occurs in many practical situations, derivation of simplified and transparent formulae for the R/T coefficients have attracted attention of many seismologists. For a brief history of the problem see, e.g., Ruger and Tsveikin (1997), Vavryčuk and Pšenčík (1998).

Vavryčuk and Pšenčík (1998) and Pšenčík and Vavryčuk (1998) derived approximate formulae for PP R/T coefficients for weak contrast interfaces separating weakly anisotropic media of arbitrary symmetry. In the final formulae, they considered a medium with P- and S-wave velocities \( \alpha^2 = A_{33} \) and \( \beta^2 = A_{55} \) as an isotropic background in each of the halfspaces surrounding the reflector. Symbols \( A_{i\ell j} \) denote density normalized elastic parameters in the Voigt notation. The above assumption concerning the background reduces applicability of the derived formulae. The formulae, which we present in this contribution, allow use of arbitrary values of \( \alpha \) and \( \beta \) for the background media. Due to the arbitrary background, the formulae contain new weak anisotropy (WA) parameters (WA parameters are generalizations of parameters introduced by Thomsen, 1986), important for characterization of the medium.

Introduction of an arbitrary background to the formulae of Vavryčuk and Pšenčík (1998) and Pšenčík and Vavryčuk (1998) has an important consequence. The formulae can be used even in cases when parameters characterizing vertical propagation, i.e., \( A_{33}, A_{44} \) or \( A_{55} \), are not known. In fact, the presented formulae can be used for retrieving these parameters from the R/T coefficients in the form of newly introduced WA parameters \( \epsilon_z \) (related to \( A_{33} \)), \( \gamma_z \) (related to \( A_{55} \)) or \( \gamma_y \) (related to \( A_{44} \)). Introduction of an arbitrary background causes appearance of the WA parameter \( \gamma_z = \gamma_y \) related to shear wave propagation, in the formula for a VTI medium.

We present two formulae for the PP reflection coefficient. One is equivalent to the formula derived by Vavryčuk and Pšenčík (1998), the other is its linearized version. The correction term due to anisotropy in the former formula is independent of the S-wave velocity \( \beta \) in the background medium. The correction term in the latter formula depends on the parameter \( \beta \) and, due to the linearization, the formula is generally slightly less accurate. Nevertheless, in this study, we concentrate on the latter formula because it is this formula, which reduces to the well-known formulae for anisotropic media of higher symmetry. With the exception of the definition of two WA parameters, the latter formula for the PP reflection coefficient reduces to the formula derived by Ruger (1997, 1998) for symmetry planes of anisotropic media of VTI, HTI or orthorhombic symmetry. We discuss the role of the above mentioned WA parameters different from Ruger's definition and show how a different definition can affect accuracy of the approximate formulae. We illustrate this on two typical models proposed by Ruger (1997). We calculate approximate reflection coefficients for angles of incidence from 0° to 40° and for all azimuths. By comparing them with exact coefficients, we show an excellent performance of the presented formulae. Anisotropy and velocity contrast are around 10% in this case. In order to show performance of the formulae in more complicated models with stronger anisotropy and velocity contrast, we calculate \( R_{pp} \) coefficients for angles of incidence between 10° and 30° and for all azimuths in a model consisting of a VTI overburden over an orthorhombic reflecting medium. Anisotropy is about 25% and a minimum velocity contrast is 13% in this case. Even in such an extreme case, the approximate formula yields results of a reasonable accuracy within the interval of angles of incidence considered. Finally, we use again Ruger's models to investigate the influence of the choice of background on the accuracy of the approximate \( R_{pp} \) coefficients.

The accuracy and form of the formulae presented (sum of an approximate R/T coefficient for weak contrast background isotropic media and a correction term depending linearly on WA parameters) offer a straightforward application. The formulae can be directly used for an inversion of observed coefficients into WA parameters, see e.g. Vavryčuk (1999). For this purpose, it is good to know the sensitivity of the formulae to WA parameters. We present maps of sensitivity of the PP reflection coefficients to most important WA parameters or their combinations in order to indicate which parameters and with which reliability can be found by inverting an \( R_{pp} \) coefficient.
2. WEAK CONTRAST PP REFLECTION COEFFICIENT FOR WEAK ANISOTROPY OF ARBITRARY SYMMETRY

An approximate formula for the PP reflection coefficient for a weak contrast interface separating two weakly anisotropic media of arbitrary symmetry, expressed with respect to an arbitrary background, is:

\[ R_{PP}(\varphi, \theta) = R_{PP}^{iso}(\theta) + \frac{1}{2\alpha^2} \left\{ \Delta(\alpha^2 \varepsilon_z) + \left[ \Delta(\alpha^2 \delta_x) - 8\Delta(\beta^2 \gamma_y) \right] \cos^2 \varphi \right. \]

\[ + \left[ \Delta(\alpha^2 \delta_y) - 8\Delta(\beta^2 \gamma_x) \right] \sin^2 \varphi + \right. \]

\[ + 2\Delta(\alpha^2 \chi_z) - 4\Delta(\beta^2 \epsilon_{45}) \cos \varphi \sin \varphi - \Delta(\alpha^2 \varepsilon_x) \right\} \sin^2 \theta + \]

\[ + \left[ \Delta(\alpha^2 \varepsilon_y) \cos^4 \varphi + \Delta(\alpha^2 \delta_z) \sin^4 \varphi + \Delta(\alpha^2 \delta_z) \cos^2 \varphi \sin^2 \varphi + \right. \]

\[ + 2\Delta(\alpha^2 \epsilon_{16}) \cos^2 \varphi + \Delta(\alpha^2 \epsilon_{26}) \sin^2 \varphi \sin \varphi \cos \varphi \right\} \sin^2 \theta \tan^2 \theta \]  

Here \( R_{PP}^{iso}(\theta) \) denotes the weak contrast PP reflection coefficient for isotropic media

\[ R_{PP}^{iso}(\theta) = \frac{1}{2} \frac{\Delta Z}{Z} + \frac{1}{2} \left[ \frac{\Delta \alpha}{\alpha} - 4 \frac{\beta^2}{\alpha^2} \frac{\Delta G}{G} \right] \sin^2 \theta + \frac{1}{2} \frac{\Delta \alpha}{\alpha} \sin^2 \theta \tan^2 \theta , \]  

where

\[ Z = \rho \alpha, \ G = \rho \beta^2. \]

The symbols \( \Delta \) denote contrasts across an interface, e.g. \( \Delta Z = Z_2 - Z_1 \). Bar over a symbol denotes an average, e.g. \( \bar{Z} = 1/2(Z_1 + Z_2) \). The index 1 corresponds to the upper halfspace, the index 2 to the lower halfspace. The symbols \( \theta \) and \( \varphi \) denote phase angles: \( \theta \) is the angle of incidence (\( \theta = 0^\circ \) for a vertical incidence) and \( \varphi \) is the azimuth (\( \varphi = 0^\circ \) for direction along \( x_1 \) axis). The reflection coefficient (1) depends on 9 of the following 15 P-wave WA parameters:

\[ \delta_x = \frac{A_{15} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2}, \]

\[ \chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2}. \]

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\[ \varepsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \varepsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \varepsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \varepsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \varepsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \varepsilon_{35} = \frac{A_{35}}{\alpha^2}, \]

\[ \varepsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \varepsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \varepsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2}, \]

see Pšenčík and Gajewski (1998), and on 3 S-wave WA parameters

\[ \gamma_x = \frac{A_{45} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{45}}{\beta^2}. \]

Here, \( \alpha \) and \( \beta \) are \( P \)- and \( S \)-wave velocities of the background isotropic medium respectively, which can be chosen arbitrarily (for example so that one or two of the WA parameters vanish).

Eqs. (1) and (2) correspond exactly to Eq. (39) of Vavryčuk and Pšenčík (1998). In Eq. (1), the WA parameters (4) and (5) are used instead of elastic parameters. Let us note that the correction term in Eq. (1), i.e., the term behind \( R_{iso}^{pp}(t) \), is independent of the parameter \( \beta \). The WA parameters appear in Eq. (1) in terms of the type \( \Delta(c^2 \varepsilon) \), where \( c \) stands for the background velocity (\( \alpha \) or \( \beta \)) and \( \varepsilon \) for any of the WA parameters. The terms \( \Delta(c^2 \varepsilon) \) can be linearized to yield

\[ \Delta(c^2 \varepsilon) \sim c^2 \Delta \varepsilon, \]

where the second-order term \( \varepsilon \Delta c^2 \) is neglected. After the linearization, Eq. (1) transforms into

\[ R_{pp}(\varphi, \theta) = R_{iso}^{pp}(\theta) + \frac{1}{2} \Delta \varepsilon_x + \frac{1}{2} \left[ \left( \Delta \delta_x - \frac{8}{\alpha^2} \Delta \gamma_x \right) \cos^2 \varphi + \left( \Delta \delta_y - \frac{8}{\alpha^2} \Delta \gamma_y \right) \sin^2 \varphi + \right. \]

\[ + 2 \left( \Delta \chi_x - \frac{8}{\alpha^2} \Delta \chi_{45} \right) \cos \varphi \sin \varphi - \Delta \varepsilon_x \right] \sin^2 \theta + \frac{1}{2} \left[ \Delta \varepsilon_x \cos^4 \varphi + \Delta \varepsilon_y \sin^4 \varphi + \right. \]

\[ + \Delta \delta_x \cos^2 \varphi \sin^2 \varphi + 2(\Delta \epsilon_{16} \cos^2 \varphi + \Delta \epsilon_{26} \sin^2 \varphi) \sin \varphi \cos \varphi \sin^2 \theta \tan^2 \theta. \]

The correction term in Eq. (7) is no more independent of \( \beta \) and due to the linearization (6), it is generally less accurate. As we shall see further, it is the linearized equation (7), which reduces to the well-known formulae, for anisotropic media of higher symmetry.

For orthorhombic media with symmetry axes coinciding with coordinate axes in both halfspaces, we have

\[ \chi_x = \chi_y = \chi_z = \epsilon_{15} = \epsilon_{16} = \epsilon_{24} = \epsilon_{26} = \epsilon_{34} = \epsilon_{35} = 0. \]

For such a specification, Eq. (7) reduces to a simple form.
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If we specify the formula (9) for the plane \((x_1, x_3)\), i.e. for \(\varphi = 0^\circ\), and for the plane \((x_2, x_3)\), i.e. for \(\varphi = 90^\circ\), and if we set \(A_2 = A_{33}\) and \(B_2 = A_{55}\), the results can be compared with corresponding formulae (22) and (24) of Ruger (1998). It can be easily shown that the formulae coincide, and that only definitions of some of the WA parameters differ. Ruger (1998) uses the WA parameter \(\delta\) introduced by Thomsen (1986)

\[
\delta = \frac{(A_{13} + A_{55})^2 - (A_{33} - A_{55})^2}{2A_{33}(A_{33} - A_{55})}.
\]

The corresponding parameter in Eq.(4) has the form

\[
\delta_x = \frac{A_{13} + 2A_{55} - A_{33}}{A_{33}}.
\]

As shown by Sayers (1994), the difference between the above two parameters is of second order. Thomsen’s \(\delta\) can be expressed in terms of \(\delta_x\) in the following way:

\[
\delta = \delta_x + \frac{1}{2} \frac{\alpha^2}{\alpha^2 - \beta^2} \delta_x^2.
\]

We can see that Thomsen’s \(\delta\) is always larger than \(\delta_x\). By neglecting the second order term in Eq.(11), we make an error of roughly 75\(\delta_x\)% assuming \(\alpha^2 - 3\beta^2\). Thus substitution of \(\delta\) by \(\delta_x\) for \(\delta_x \approx \pm 0.1\) leads to errors in \(\delta\) of about \(\pm 7.5\%). Although the use of \(\delta\) instead of \(\delta_x\) in the formulae for the reflection coefficients leads to slightly more accurate results for some angles of incidence, its use in the above equations would be artificial. Therefore, the WA parameters \(\delta_x\), \(\delta_y\) and \(\delta_z\) related linearly to elastic parameters, see Eq.(4), are used in all calculations in this paper.

In his Eq.(23), Ruger (1998) expresses the approximate \(R_{PP}\) coefficient in the plane \((x_1, x_3)\) in terms of the shear wave splitting parameter \(\gamma\). His \(\gamma\) is defined as

\[
\gamma = \frac{A_{44} - A_{55}}{2A_{55}},
\]

while our Eqs.(5) yield for Ruger’s specification \(\beta^2 = A_{44}\)
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\[ \gamma_x = \frac{A_{55} - A_{44}}{2A_{44}} . \]  

Ruger's \( \gamma \) must be compared with \(-\gamma_x\); \( \gamma \) expressed in terms of \(-\gamma_x\) reads

\[ \gamma \sim -\gamma_x + 2\gamma_x^2 . \]  

We can see that neglecting the second order term in (14) can lead to an error of roughly 200\( \gamma_x \)%.

For \( \gamma_x \sim 0.1 \), substitution of \( \gamma \) by \( \gamma_x \) leads to an error substantially larger than in substituting \( \delta \) for Thomsen's \( \delta \). The use of \( \gamma \) defined in (12) in the approximate formula for PP-wave reflection coefficient causes 'reduced numerical accuracy' reported by Ruger (1998). Use of \( \gamma \) and \( \gamma_x \), defined in (5), in Eq.(9) removes this problem.

For an isotropic upper halfspace, which we consider in some of the following examples, the contrasts of the WA parameters reduce to the parameters of the lower halfspace: \( \Delta \delta_x \rightarrow \delta_x \), \( \Delta \varepsilon_x \rightarrow \varepsilon_x \), etc., where \( \delta_x \), \( \varepsilon_x \), etc. denote the WA parameters of the lower halfspace. If the lower halfspace is transversely isotropic with vertical axis of symmetry (VTI), the WA parameters have, in addition to (8), the following symmetries:

\[ \delta_y = \delta_x , \ \delta_z = 2\varepsilon_x , \ \varepsilon_y = \varepsilon_x , \ \gamma_y = \gamma_x . \]  

In this case, Eq.(7) for the weak contrast reflection coefficient attains the form:

\[ R_{PP}(\theta) = R_{PP}^{iso}(\theta) + \frac{1}{2} \varepsilon_x + \frac{1}{2} \left[ \Delta \delta_x - 8 \frac{\beta^2}{c^2} \gamma_x \right] \sin^2 \theta + \frac{1}{2} \varepsilon_x \sin^2 \theta \tan^2 \theta . \]  

The symbol \( R_{PP}^{iso}(\theta) \) denotes again the weak contrast PP reflection coefficient for isotropic media (2).

We can see that the coefficient (16) differs from the coefficient derived for the same model by Ruger (1997), see also Thomsen (1993). There are two WA parameters, \( \varepsilon_x \) and \( \gamma_x \), which do not appear in Ruger's formula. The reason for the appearance of these parameters is that Ruger (1997) considers background velocities specified as \( \alpha^2 = A_{33}, \beta^2 = A_{55} \). This leads to \( \varepsilon_x = 0 \) and \( \gamma_x = 0 \) and thus to reduction of (16) to Ruger's formula.

Another slight difference consists in a different definition of Thomsen's \( \delta \) and our \( \delta_x \), (see Eqs.(4)), which has been discussed above. From comparison of formula (16) with Ruger's (1997) formula, we can see the advantage of using an arbitrary background. Eq.(16) allows, in principle, the determination of \( \varepsilon_x \), and thus of \( A_{33} \), and of \( \gamma_x \), and thus of \( A_{55} \), even in VTI media.

For an isotropic halfspace over a halfspace with transversely isotropic medium with horizontal axis of symmetry (HTI), parallel to the axis \( x_1 \), the WA parameters of the lower halfspace satisfy, in addition to (8), the following relations:

\[ \delta_y = 2\varepsilon_x , \ \delta_z = \delta_x , \ \varepsilon_y = \varepsilon_x . \]  

Eq.(7) then yields
Comparison of Eq. (18) with Eq. (5) of Rüger (1998) leads to a similar discussion as that following Eq. (9).

We have introduced the specialized versions of Eq. (7) only for comparison with formulae derived for special situations by other authors. In applications, use of the formulae (1) or (7) is preferable.

3. WEAK CONTRAST PP TRANSMISSION COEFFICIENT FOR WEAK ANISOTROPY OF ARBITRARY SYMMETRY

The approximate formula for the PP transmission coefficient for a weak contrast interface separating weakly anisotropic media of arbitrary symmetry, expressed with respect to an arbitrary background, has the following form

\[
T_{PP}(\varphi, \theta) = T_{PP}^{iso}(\theta) - \frac{1}{2} \Delta e_x + \frac{1}{2} \left[ \Delta \delta_x \cos^2 \varphi + \Delta \delta_y \sin^2 \varphi + 2 \Delta \chi_z \cos \varphi \sin \varphi - \Delta e_z \right] \times \\
\times \sin^2 \theta + \frac{1}{2} (\epsilon_x \cos^4 \varphi + \epsilon_z \sin^4 \varphi + \delta_x \sin^2 \varphi \cos^2 \varphi) \sin^2 \theta \tan^2 \theta . \tag{18}
\]

Here, \( T_{PP}^{iso}(\theta) \) denotes the weak contrast PP transmission coefficient for isotropic media

\[
T_{PP}^{iso}(\theta) = 1 - \frac{1}{2} \frac{\Delta Z}{Z} + \frac{1}{2} \frac{\Delta \alpha}{\alpha} \tan^2 \theta . \tag{20}
\]

The approximate formula (19) includes already the linearization (6). It depends on all 15 P-wave WA parameters. It, however, does not depend on parameters related to shear wave propagation, see Eqs. (5). Thus, from the transmission coefficient, all the parameters...
necessary for the determination of the $P$-wave phase velocity (see Eq. (17a) of Pšenčík and Gajewski, 1998) can be, in principle, recovered. On the other hand, no estimates of parameters related to shear wave propagation can be made.

For orthorhombic media with symmetry axes coinciding with coordinate axes in both halfspaces, Eqs. (8) hold and Eq. (19) reduces to

$$ T_{PP}(\varphi, \theta) = T_{PP}^{00}(\theta) - \frac{1}{2} \Delta \varepsilon_z + \frac{1}{2} \left[ \Delta \delta_x \cos^2 \varphi + \Delta \delta_y \sin^2 \varphi - \Delta \varepsilon_z \right] \sin^2 \theta + $$

$$ + \frac{1}{2} \left[ \Delta \varepsilon_x \cos^4 \varphi + \Delta \varepsilon_x \sin^4 \varphi + \Delta \delta_z \cos^2 \varphi \sin^2 \varphi \right] \sin^2 \theta \tan^2 \theta + $$

$$ + \left[ \Delta \varepsilon_x \cos^4 \varphi + \Delta \varepsilon_x \sin^4 \varphi + \Delta \delta_z \cos^2 \varphi \sin^2 \varphi - \Delta \delta_z \cos^2 \varphi \right] \sin^2 \theta$$

$$ - \Delta \delta_x \sin^2 \varphi + \Delta \varepsilon_z \right] \sin^4 \theta. \quad (21) $$

Again, all six WA parameters necessary for the determination of the $P$ wave phase velocity from Eq. (21) of Pšenčík and Gajewski (1998) can be, in principle, recovered from Eq. (21).

For an isotropic upper halfspace and transversely isotropic lower halfspace with vertical axis of symmetry (VTI), i.e. for the WA parameters specified as in Eqs. (8) and (15), formula (19) substantially simplifies. It reads

$$ T_{PP}(\varphi, \theta) = T_{PP}^{00}(\theta) - \frac{1}{2} \varepsilon_z + \frac{1}{2} (\delta_x - \varepsilon_z) \sin^2 \theta + \frac{1}{2} \varepsilon_x \sin^2 \theta \tan^2 \theta + $$

$$ + (\varepsilon_x + \varepsilon_z - \delta_z) \sin^4 \theta. \quad (22) $$

For an isotropic upper halfspace over a halfspace with transversely isotropic medium with horizontal axis of symmetry, parallel to the axis $x_1$, i.e. for the WA parameters specified as in Eqs. (8) and (17), the formula (19) reduces to

$$ T_{PP}(\varphi, \theta) = T_{PP}^{00}(\theta) - \frac{1}{2} \varepsilon_z + \frac{1}{2} (\delta_x \cos^2 \varphi - \varepsilon_z \cos 2\varphi) \sin^2 \theta + $$

$$ + \frac{1}{2} (\varepsilon_x \cos^4 \varphi + \varepsilon_z \sin^4 \varphi + \delta_x \sin^2 \varphi \cos^2 \varphi) \sin^2 \theta \tan^2 \theta + $$

$$ + (\varepsilon_x + \varepsilon_z - \delta_x) \cos^4 \varphi \sin^4 \theta. \quad (23) $$
4. ACCURACY OF APPROXIMATE FORMULAE

In this section, we illustrate the accuracy of the formula (7) for the approximate reflection coefficient. For this purpose, we use models 6 and 11 of Ruger (1997). From the contrasts of P-wave velocity, impedance and shear wave modulus and from Thomsen's parameters specified in Ruger (1997), we reconstructed elastic parameters of his models. The isotropic material in the upper halfspace of model 6 is specified by $\alpha = 2.26$ km/s, $\beta = 1.43$ km/s and $\rho = 2.70$ g/cm$^3$, and in model 11 by $\alpha = 2.76$ km/s, $\beta = 1.58$ km/s and $\rho = 2.70$ g/cm$^3$. The lower halfspace is the same in both cases and it is specified again by $\rho = 2.70$ g/cm$^3$ and by the following matrix of density normalized elastic parameters (in km$^2$/s$^2$) of an HTI medium:

\[
\begin{pmatrix}
5.00 & 1.82 & 1.82 & 0.00 & 0.00 & 0.00 \\
6.25 & 1.75 & 0.00 & 0.00 & 0.00 \\
6.25 & 0.00 & 0.00 & 0.00 \\
2.25 & 0.00 & 0.00 \\
1.87 & 0.00 \\
1.87
\end{pmatrix}
\]

The axis of symmetry of the HTI medium is parallel to the $x$-axis and corresponds to azimuth of 0°. P-wave anisotropy is less than 10%. P-wave velocity contrast is about 10% for vertical incidence. For increasing angle of incidence in the symmetry plane (plane $(x_1, x_3)$), the contrast decreases in model 6 and increases in model 11. Thus we should expect a better fit of approximate (Eq.(7)) and exact coefficients in the symmetry plane in model 6 than in model 11. The contrast remains constant in the isotropy plane (plane $(x_2, x_3)$).

In Figure 1 we show a comparison of exact and approximate reflection coefficients in the symmetry plane (0°) and in the isotropy plane (90°) for model 6 (upper plot) and model 11 (bottom plot). Since the approximation works only in the subcritical region, we deal with real-valued coefficients. The approximate coefficients were calculated using Eq.(7) with $\alpha^2 = A_{33}, \beta^2 = A_{44}$, which implies $e_z = 0$ and $G_y = 0$.

The WA parameter $\gamma_x$ was defined successively as $\gamma$ in Eq.(12) and as $\gamma_x$ in Eq.(13). The former specification corresponds to Ruger's (1997, 1998) definition, the latter specification corresponds to the definition of the WA parameters introduced in this paper. Since in the isotropy plane the approximate formula does not depend on the WA parameter $\gamma_x$, the curves calculated according to Eqs.(12) and (13) coincide. In the symmetry plane, however, we can observe considerable differences between curves calculated according to Eqs.(12) and (13). The curves with $\gamma_x$ specified by Eq.(13) yield very good approximation of exact curves even for larger angles of incidence while the curves corresponding to $\gamma_x$ specified as $\gamma$ in Eq.(12) start to deviate from exact ones already for small angles of incidence.
Fig. 1. Comparison of exact and approximate $R_{pp}$ coefficients in model 6 (upper plot) and model 11 (bottom plot), in the symmetry plane ($\varphi = 0^\circ$) and in the isotropy plane ($\varphi = 90^\circ$). The approximate results are obtained with $\gamma$ parameters defined in Eqs. (12) and (13). Specification by Eq. (12) leads to considerable inaccuracies in the symmetry plane.
Figures 2 and 3 show that applicability of formula (7) for the $R_{pp}$ coefficients is not limited to only planes of symmetry and isotropy. The figures show the coefficients as a function of angle of incidence (horizontal axes) and azimuth (vertical axes). As a background we have chosen $a^2 = A_{33}$ and $b^2 = A_{44}$ as in Rüger (1997). The left figure shows exact results and the middle figure shows results of the approximate formula. The right figure shows the difference between approximate and exact coefficients. The azimuths range from 0° to 90° and the angles of incidence from 0° to 40°. Because anisotropy of the models used is not too strong and the velocity contrast across the interface is mild, the figures show a very good performance of approximate formulae although the coefficients show relatively strong azimuthal variation.

In order to illustrate performance of the approximate formula (7) for a stronger anisotropy and stronger velocity contrast, we show comparison of exact and approximate calculations for a model proposed by Neves and de Hoop (2000) in Figure 4. Their model consists of two anisotropic halfspaces. The upper halfspace is VTI and it represents a shale. Its density is 2.5 g/cm$^3$. Its matrix of density normalized elastic parameters (in km$^2$/s$^2$) has the form:

$$
\begin{pmatrix}
6.94 & 3.64 & 2.70 & 0.00 & 0.00 & 0.00 \\
6.94 & 2.70 & 0.00 & 0.00 & 0.00 \\
4.28 & 0.00 & 0.00 & 0.00 \\
1.23 & 0.00 & 0.00 \\
1.23 & 0.00 \\
1.65 &
\end{pmatrix}
$$

The lower halfspace is orthorhombic and it represents a fractured sandstone. Its density is 2.2 g/cm$^3$ and the matrix of density normalized elastic parameters has the form (in km$^2$/s$^2$):

$$
\begin{pmatrix}
12.27 & 4.87 & 3.05 & 0.00 & 0.00 & 0.00 \\
13.44 & 3.31 & 0.00 & 0.00 & 0.00 \\
8.10 & 0.00 & 0.00 & 0.00 \\
2.70 & 0.00 & 0.00 \\
2.18 & 0.00 \\
2.97 &
\end{pmatrix}
$$

From the above matrices we can see that $P$-wave anisotropy ($\overline{(v_{\text{max}} - v_{\text{min}})v_{\text{aver}}}$) in both halfspaces is rather strong (24% and 25%) so that neither of the halfspaces can be called weakly anisotropic. The $P$-wave velocity contrast is also rather strong (for vertical incidence, for which it is smallest, it reaches 13%). For such a model, we can expect only limited applicability of Eq.(7). The situation is complicated even more by a shear wave singularity in VTI medium for nearly normal incidence, which makes calculation of exact reflection coefficients difficult. Despite all these unfavorable effects, the approximate formula yields reasonably good approximation of exact results for small angles of incidence, see Figure 4. The azimuths range again from 0° to 90° and the angles of incidence from 10° to 30°. The range between 0° and 10° was omitted in order to avoid problems caused in calculation of exact coefficients by a shear wave singularity. The approximate formula had no such problems.
Fig. 2. The maps of exact and approximate PP reflection coefficients (left and middle frames) and absolute errors of the approximate coefficient (right frame) in model 6. Background: $\alpha^2 = A_{33}$, $\beta^2 = A_{44}$. P-wave anisotropy and velocity contrast about 10%.
Fig. 3. The maps of exact and approximate PP reflection coefficients (left and middle frames) and absolute errors of the approximate coefficient (right frame) in model 1. Background: $\varphi = A_{33}$, $\beta^2 = A_{44}$, P-wave anisotropy and velocity contrast about 10%.
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Fig. 4. The maps of exact and approximate PP reflection coefficients (left and middle frames) and absolute errors of the approximate coefficient (right frame) in model of a VTI halfspace over an orthorhombic halfspace. Background: $\varepsilon = A_3 = A_5$. $P$-wave anisotropy of about 25%, $P$-wave velocity contrast more than 13%.
In this section, we illustrate the influence of the choice of the isotropic background on the quality of the approximation of Eq.(7). For this purpose, we use again models 6 and 11 of Ruger (1997) specified in Sec. 4. For both models, we calculated differences between approximate and exact $R_{PP}$ coefficients for varying specifications of the background for angles of incidence between 0° and 40°, for the planes of symmetry and isotropy. In Figure 5, differences for the model 6 in the plane of symmetry (top) and in the plane of isotropy (bottom) are shown. The isotropic background in the reflecting medium was chosen as follows: $\alpha^2 = A_{33}$, i.e. $\alpha = 2.5$ km/s and $\beta$ attained successively values of 1.32, 1.37, 1.40, 1.47, 1.50 and 1.54 km/s. The value of 1.37 km/s corresponds approximately to $(A_{55})^{1/2}$ and the value of 1.50 km/s to $(A_{44})^{1/2}$. Values of the $R_{PP}$ coefficient vary from 0.05 to 0.075. We can see that the approximate formula works better in the symmetry plane than in the isotropy plane. For most backgrounds the errors remain below 10%. Neither $A_{44}$ nor $A_{55}$ yields the best approximation. Background $A_{44}$ gives better results in the isotropy plane ($A_{44}$ is related to the SH wave propagation), background $A_{55}$ in the symmetry plane ($A_{55}$ is related to the SV wave propagation). The best approximation in the symmetry plane is obtained for values of $\beta^2$ between $A_{44}$ and $A_{55}$.

Figure 6 shows results for the model 11 in the same display as in Figure 5. Values of the $R_{PP}$ coefficient are now between $-0.05$ and $-0.035$. Due to the increasing contrast with increasing angle of incidence in the symmetry plane, the accuracy of the approximate coefficient rapidly decreases. For larger angles of incidence, the approximate formula yields results with a considerable error. Up to 20°, however, the errors are well below 10% for all the backgrounds.

The above two figures seem to indicate that the best choice of $\beta^2$ is somewhere between $A_{44}$ and $A_{55}$, probably closer to $A_{44}$. Since $\beta$ does not appear in the AVO intercept term in Eq.(7) and $\alpha^2$ was specified as $A_{33}$, all the difference curves start from zero for the vertical incidence, which indicates that for this case, the approximate coefficient coincides with the exact one. Let us remind that the dependence of the correction term due to anisotropy in the expression for the approximate coefficient on the parameter $\beta$ is property of the formula (7). The correction term in Eq.(1) is independent of $\beta$.

Let us now specify $\beta = 1.47$ km/s and let us vary $\alpha$. We choose a successively as 2.00, 2.25, 2.50, 2.75 and 3.00 km/s. The results for the model 6 are shown in Figure 7, in the same form as in Figures 5 and 6. We can see that variation of $\alpha$ introduces a shift of the approximate coefficient with respect to the exact one even for vertical incidence. For the choices of the background not too different from $A_{33}$, the shift of the coefficient varies only little, especially for small angles of incidence. We can see that a 10% difference of $\alpha^2$ from $A_{33}$ introduces approximately 10% shift of the approximate coefficient for practically all considered angles of incidence. This means that the gradient term in Eq.(7) is only weakly dependent on the choice of $\alpha$. As could be expected, the best approximation is obtained for the choice $\alpha^2 = A_{33}$, i.e. for the choice commonly used in seismic practice. It is of interest to check the curve corresponding to $\alpha = 2.25$ km/s. This curve corresponds roughly to a homogeneous continuous background (the $P$- and $S$-wave velocity contrasts are negligible in this case). Similar features can be observed in Figure 8, which shows the same as Figure 7 but for model 11.
Fig. 5. Deviations $\Delta R_{pp}$ of approximate $R_{pp}$ coefficients (Eq.(7)) from exact ones in model 6 in the symmetry plane (top) and in the isotropy plane (bottom). Individual curves correspond to different $S$-wave velocities $\beta$ of the background. $P$-wave velocity is specified as $a^2 = A_{33} = 6.25 \text{ (km/s)}^2$. 

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Fig. 6. Deviations $\Delta R_{pp}$ of approximate $R_{pp}$ coefficients (Eq. (7)) from exact ones in model 11 in the symmetry plane (top) and in the isotropy plane (bottom). Individual curves correspond to different S-wave velocities $\beta$ of the background. P-wave velocity is specified as $\alpha^2 = \lambda_{33} = 6.25$ (km/s)$^2$. 
Fig. 7. Deviations $\Delta R_{pp}$ of approximate $R_{pp}$ coefficients (Eq.(7)) from exact ones in model 6 in the symmetry plane (top) and in the isotropy plane (bottom). Individual curves correspond to different $P$-wave velocities $A$ of the background. $S$-wave velocity is specified as $B = 1.47$ km/s.
Fig. 8. Deviations $\Delta R_{pp}$ of approximate $R_{pp}$ coefficients (Eq.(7)) from exact ones in model 11 in the symmetry plane (top) and in the isotropy plane (bottom). Individual curves correspond to different $P$-wave velocities $a$ of the background. $S$-wave velocity is specified as $\beta = 1.47$ km/s.
For comparison with results in Figure 7, deviations of approximate \( R_{pp} \) coefficients calculated from Eq.(1) from exact ones are shown in Figure 9. We can see a considerable reduction of deviations and their nearly symmetric distribution with respect to the curve corresponding to \( \alpha^2 = A_{33} \). The slope of the individual curves is effectively zero, which implies that the gradient term in Eq.(7) is effectively independent of the choice of \( \alpha \).

6. SENSITIVITY TO WA PARAMETERS

Comparison of approximate formulae (7) for the \( PP \) reflection coefficient and (19) for the \( PP \) transmission coefficient implies that the formula for the transmission coefficient contains more information about WA parameters. The transmission coefficient depends on all the \( P \)-wave WA parameters. Thus it could be useful to use transmission coefficients in the AVO studies connected with VSP experiments. Nevertheless, presently, the main source of information about the parameters of the medium in AVO/AVA studies are the reflection coefficients. It is generally accepted that the determination of the multiplicative factor of \( \sin^2 \theta \tan^2 \theta \) in the expression for the reflection coefficient is difficult (Rüger and Tsvankin, 1997). For practical purposes, only the remaining intercept and gradient terms can be used. These terms depend on only four combinations of WA parameters. Sensitivity of the approximate reflection coefficient formula to these four parameters is shown in Figure 10. Sensitivity of a parameter is the angle-dependent coefficient of this parameter in Eq.(7). The reflection coefficient is sensitive to \( \Delta \varepsilon_z \) (Figure 10a) equally well for all azimuths. Sensitivity to terms \( \Delta \delta_x - 8 \frac{\beta^2}{\alpha^2} \Delta \gamma_x \) (Figure 10b) and \( \Delta \delta_y - 8 \frac{\beta^2}{\alpha^2} \Delta \gamma_y \) (Figure 10c) is similar. In both cases it increases with increasing angle of incidence. In the former case, it is large along the \( x_1 \)-axis and zero along the \( x_2 \)-axis, in the latter case, it is opposite. Sensitivity to \( \Delta \chi_z - 4 \frac{\beta^2}{\alpha^2} \Delta \varepsilon_{45} \) (Figure 10d) also increases with increasing angle of incidence but it is zero along the \( x_1 \)- and \( x_2 \)-axes and maximum for azimuths of 45° and 135°.

To separate the WA parameters in the above mentioned terms, additional independent information is necessary. \( P \)-wave moveout velocity can serve for this purpose (Rüger and Tsvankin, 1997). Equation (43) of Psencík and Gajewski (1998) yields such an independent information on the WA parameters \( \delta_x, \delta_y, \chi_z \). With these parameters known, \( \gamma_c, \gamma_t \) and \( \varepsilon_{45} \) can be determined easily.
Fig. 9. Deviations $\Delta R_{pp}$ of approximate $R_{pp}$ coefficients (Eq.(1)) from exact ones in model 6 in the symmetry plane (top) and in the isotropy plane (bottom). Individual curves correspond to different P-wave velocities $\alpha$ of the background. S-wave velocity is specified as $\beta = 1.47$ km/s.
Fig. 10. Sensitivity maps of the $R_{PP}$ coefficient with respect to terms (a) $\Delta \varepsilon_z$, (b) $\Delta \delta_x = -8 \frac{\beta^2}{\alpha^2} \Delta \gamma_x$, (c) $\Delta \delta_y = -8 \frac{\beta^2}{\alpha^2} \Delta \gamma_y$ and (d) $\Delta \chi_z = \frac{4 \beta^2}{\alpha^2} \Delta \varepsilon_{45}$. 

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7. CONCLUSIONS


We presented two formulae, (1) and (7) for the approximate PP reflection coefficient. The formula (1) corresponds exactly to the formula derived by Vavryčuk and Pšenčík (1998). The correction term due to anisotropy in this formula is independent of the choice of the S-wave velocity $\beta$ of the background media. The formula (7), obtained by linearization with respect to the WA parameters, is generally slightly less accurate and its correction term depends on the choice of $\beta$. The latter formula reduces to the formulae for anisotropic media of higher symmetry known from literature. Therefore, we concentrated on the formula (7) in this study.

Experiments with various backgrounds have shown that for larger angles of incidence the formula (7) depends rather strongly on the choice of the parameter $\beta$ if this parameter is not chosen between $A_{44}$ and $A_{55}$. The intercept term of formula (7) depends significantly on the parameter $\alpha$. The gradient term, however, depends on the choice of $\alpha$ only slightly. Thus it could be, at least theoretically, used for retrieving the WA parameters (or their combinations) affecting the gradient term, independently of the choice of $\alpha$. One of the WA parameters is the parameter $e_z$, from which the elastic parameter $A_{33}$ (commonly assumed to be known) can be found. The fact that the correction term in formula (1) does not depend on $\beta$ and results of Figure 9 indicate that the specification of a coefficient in the form shown in Eq.(1) could be more convenient for inversion.

By specifying the general formulae for anisotropic media with a higher symmetry, we found that reduced formulae differ from formulae derived for special situations by other authors by a different definition of the WA parameters. It was shown, for example, that the differences in the definition of the WA parameter $\gamma_k$ can lead to a substantial decrease of accuracy of approximate formulae (see Figure 1).

We illustrated the accuracy of approximate formulae by comparing them numerically with exact coefficients. For models with anisotropy and velocity contrast about 10%, the error of approximate $R_{pp}$ formulae is around 1% in most of the considered region. For a model with a lower symmetry and stronger anisotropy (about 25%) and velocity contrast larger than 13%, the approximate $R_{pp}$ formula yields results with approximately 10% error.

Presented formulae allow simple sensitivity studies. As to the P-wave WA parameters, the approximate formula for the transmission coefficient seems to be more informative than the approximate formula for the reflection coefficient. Because of the wide practical use of the reflection coefficients in the AVO studies, we presented, however, sensitivity maps for the $R_{pp}$ coefficient.

Presented formulae for the approximate reflection/transmission coefficients can find applications not only in studying AVO/AVA in weakly anisotropic media with very
general anisotropy, but also in higher symmetry media like orthorhombic or hexagonal ones with arbitrarily oriented axes of symmetry.

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