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Acoustic and elastodynamic 3D Green's functions for isotropic media with a weak velocity gradient

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Abstract

Approximate analytical formulae for complete acoustic and elastodynamic 3D Green's functions in isotropic media with a weak and constant velocity gradient are presented. The formulae were derived by analytical calculation of higher-order approximations of the ray series. The ray series of the acoustic Green's function consists of only one non-zero term, the ray series of the elastodynamic Green's function consists of three non-zero terms including the zeroth-order term. Since the ray series is finite for both the Green's functions, the formulae are complete and valid in the whole frequency range. The formulae are approximate because we assumed a weak velocity gradient and used the first-order perturbation theory. Moreover, the formulae are valid only in a limited region around a point source. A wavefield generated by an explosive point source, and the elastostatic Green's function have also been derived.

1. Introduction

The exact and explicit analytical formulae for the acoustic and elastodynamic 3D Green's functions are known only for very few types of media. The most famous solution is the Stokes solution for a homogeneous and unbounded isotropic medium. For the acoustic Green's function, we also know a few solutions for inhomogeneous media [7,9,11,12]. For example, the exact and closed-form solution by Li et al. [11] was derived for a medium with a constant gradient of the square of slowness V^{-2}, and is expressed in terms of Airy functions and their derivatives. However, for the elastodynamic Green's function, the problem is more complex, and no explicit closed-form solution for the medium used by Li et al. [11] or for another type of continuously inhomogeneous media is known.

In this paper, we derive the acoustic as well as elastodynamic 3D Green's functions for the simplest type of a continuously inhomogeneous medium: for a medium with a *weak and constant velocity gradient*. Under the weak-gradient assumption, we can use the first-order perturbation theory, which significantly simplifies the problem [5]. We derive the complete Green's functions by calculating analytically higher-order approximations of the ray series in the asymptotic ray theory [1–4,8]. This approach has been successfully applied in calculating formulae for multipolar elastodynamic wavefields in homogeneous isotropic media [15], for the SH-wave Green's function in homogeneous transversely isotropic media [16], and for the complete elastodynamic Green's function in homogeneous weak transversely isotropic media [14]. Firstly, we derive the acoustic Green's function and show that it is

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extremely simple for the mentioned type of inhomogeneous medium. It does not differ from the Green's function in homogeneous media, only corrections of traveltimes must be considered. The same form of acoustic Green's function is also obtained by linearizing the solution by Li et al. [11]. This can be used as a check to find out if the ray-theory approach is correct. Secondly, we derive the elastodynamic Green's function and show that it is not very complicated as compared to the Green's function in the homogeneous medium. Formulae for the wavefield generated by an explosive source, and for the elastostatic Green's function are also given.

2. Acoustic and elastodynamic ray-theory Green's functions

2.1. Definition of the acoustic and elastodynamic Green's functions

Acoustic and elastodynamic Green's functions $G(\mathbf{x}, t)$ and $G_{in}(\mathbf{x}, t)$ for a perfectly elastic, inhomogeneous, isotropic medium are solutions of the following equations:

$$\frac{1}{\rho c^2}\ddot{G} - \left(\frac{1}{\rho}G_{,j}\right)_{,j} = \delta(\mathbf{x})\delta(t), \tag{1a}$$

$$\rho\ddot{G}_{in} - (\lambda + \mu)G_{jn,ij} - \mu G_{in,jj} - \lambda_{,i}\, G_{jn,j} - \mu_{,j}\, G_{in,j} - \mu_{,j}\, G_{jn,i} = \delta_{in}\delta(\mathbf{x})\delta(t), \tag{1b}$$

where $c(\mathbf{x}) = \sqrt{\lambda/\rho}$ is the acoustic velocity, $\rho(\mathbf{x})$ is the density of the medium, $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the Lamé constants, δ_{in} is the Kronecker delta, and $\delta(t)$ is the Dirac delta function. Dots over the quantities indicate the time derivative, indices after the comma denote the spatial derivative. The Einstein summation convention is applied, where repeated indices mean summation. We assume that the source is located in the origin of coordinates and the force is acting at time $t_0 = 0$.

2.2. Ray expansion of Green's functions

We seek a solution of Eqs. (1a) and (1b) in the form of the ray series ([3], Eq. (2.3)):

$$G(\mathbf{x}, t) = \sum_{K=0}^{\infty} G^{(K)}(\mathbf{x}, t) = \sum_{K=0}^{\infty} U^{(K)}(\mathbf{x}) f^{(K)}(t - \tau(\mathbf{x})), \tag{2a}$$

$$G_{in}(\mathbf{x}, t) = \sum_{K=0}^{\infty} G_{in}^{(K)}(\mathbf{x}, t) = \sum_{K=0}^{\infty} [U_{in}^{P(K)}(\mathbf{x})\, f^{(K)}(t - \tau^P(\mathbf{x})) + U_{in}^{S(K)}(\mathbf{x}) f^{(K)}(t - \tau^S(\mathbf{x}))], \tag{2b}$$

where $(\mathrm{d}/\mathrm{d}t) f^{(K)}(t) = f^{(K-1)}(t)$.

K denotes the order of the ray approximation, $G^{(K)}(\mathbf{x}, t)$ and $G_{in}^{(K)}(\mathbf{x}, t)$ are the acoustic and elastodynamic Kth-order ray-theory Green's functions, $U(\mathbf{x})$ is the acoustic ray amplitude, $U_{in}^{P}(\mathbf{x})$ and $U_{in}^{S}(\mathbf{x})$ are the elastodynamic ray amplitudes of P and S waves, $f(t)$ is the time function, $\tau(\mathbf{x})$, $\tau^P(\mathbf{x})$ and $\tau^S(\mathbf{x})$ are the traveltimes.

For determining the ray expansions (2a) and (2b) of acoustic and elastodynamic Green's functions $G(\mathbf{x}, t)$ and $G_{in}(\mathbf{x}, t)$ we must first construct the rays and calculate traveltimes $\tau(\mathbf{x})$, $\tau^P(\mathbf{x})$ and $\tau^S(\mathbf{x})$. The rays are constructed from the *ray-tracing equations* and the traveltimes are calculated by integration along the rays. Since solving the ray-tracing equations is a standard and well-known procedure studied by many authors [2,5,10,13], we shall not describe it here.

Having calculated the traveltimes, we must specify time functions $f^{(K)}(t)$ for $K \geq 0$, and calculate ray amplitudes $U^{(K)}(\mathbf{x})$, $U_{in}^{P(K)}(\mathbf{x})$ and $U_{in}^{S(K)}(\mathbf{x})$ for $K \geq 0$. The time functions are expressed very simply as

$$f^{(0)}(t) = \delta(t), \quad f^{(1)}(t) = H(t) \quad \text{and} \quad f^{(K)}(t) = \frac{t^{K-1}}{(K-1)!}H(t) \quad \text{for } K > 1. \tag{3}$$

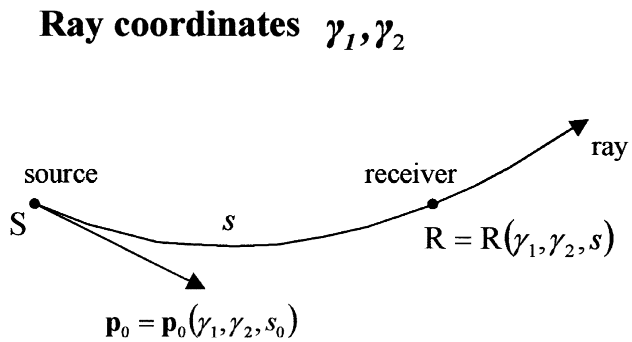

Fig. 1. Definition of ray coordinates γ_1 and γ_2. $\mathbf{p}_0$ is the initial slowness vector of the ray at the source.

The calculation of the ray amplitudes, however, is more involved. The ray amplitudes are calculated by solving the *basic ray-theory equations* (see [3], Eq. (2.10)), which form a recurrent system of equations for each Kth-order ray approximation. The calculation of the Kth-order ray amplitude involves solving the *transport equation* of the Kth-order and determining the Kth-order integration constants. The transport equation is an ordinary differential equation of the first-order and its solution is well known (see [3], Eqs. (2.25)–(2.27)). The zeroth-order integration constant can also be determined without problem. It is calculated by matching the ray solution with the far-field approximation of the Green's function for homogeneous media. However, the formulae for the higher-order integration constants of the ray-theory Green's function are lacking in the literature. Therefore, in Sections 2.3 and 2.4 we present the formulae for the higher-order ray amplitudes without specifying the higher-order integration constants. The method of determining the integration constants of the Green's functions in the medium under consideration will be described in Section 4.

2.3. The ray amplitudes of the acoustic Green's function

The ray amplitudes of the acoustic Green's function, $U^{(K)}(\mathbf{x})$, $K \geq 0$, are expressed as follows [2]:

$$\text{for } K = 0: \ U^{(0)}(\mathbf{x}) = \frac{\sqrt{\rho(s_0)\rho(s)c(s_0)c(s)}}{4\pi}\frac{1}{J(s_0, s)},$$

$$\text{for } K > 0:$$

$$U^{(K)}(\mathbf{x}) = \frac{\sqrt{\rho(s)\,c(s)}}{2J(s_0, s)}\left\{C^{(K)}(s_0, \gamma_1, \gamma_2) + \int_{s_0}^{s} \sqrt{\rho(s)'\,c(s)'}\,J(s_0, s')\,(\rho^{-1}U^{(K-1)}_{,j})_{,j}\ \mathrm{d}s'\right\}, \tag{4}$$

where γ_1, γ_2 are the ray parameters, and s is the arclength of the ray (see Fig. 1). The ray parameters γ_1, γ_2 specify the ray uniquely and can be defined, for example, as the take-off angles of the ray from the source. The integration in (4) is along the ray. The quantities with argument s_0 and s represent the values at the source and observation point, respectively. Quantity $C^{(K)}$ denotes the integration constant for the Kth-order ray approximation of the ray expansion, and quantity $J(s_0, s)$ denotes the relative geometrical spreading from the source to the observation point [2]:

$$J(s_0, s) = \sqrt{\left|\frac{\det \mathbf{Q}(s)}{\det \mathbf{P}\,(s_0)}\right|}, \quad Q_{NM}(s) = \left.\frac{\partial q_N}{\partial \gamma_M}\right|_s, \quad P_{NM}(s_0) = \left.\frac{\partial^2 \tau}{\partial q_N \partial \gamma_M}\right|_{s_0}, \ N, M = 1, 2. \tag{5}$$

Matrix $\mathbf{Q}(s)$ is the 2×2 transformation matrix from ray coordinates γ_1 and γ_2 to ray-centered coordinates q_1 and q_2 defined in the plane perpendicular to the ray (see Fig. 2). Matrix $\mathbf{P}(s_0)$ is the 2×2 transformation matrix from

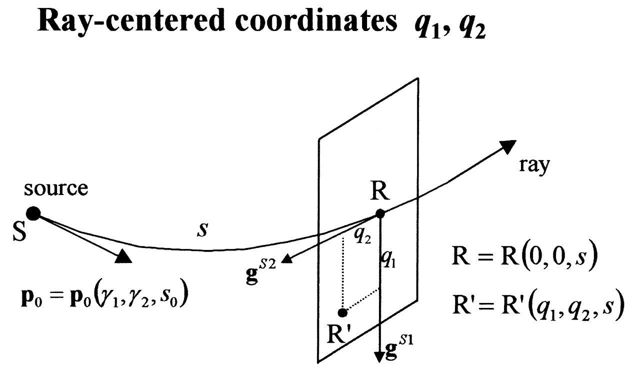

Fig. 2. Definition of the ray-centered coordinates q_1 and q_2. Vectors $\mathbf{g}^{S1}$ and $\mathbf{g}^{S2}$ are basis vectors perpendicular to the tangent to the ray.

ray coordinates γ_1 and γ_2 to the ray-centered components of slowness vector $p_1^{(q)} = \partial\tau/\partial q_1$ and $p_2^{(q)} = \partial\tau/\partial q_2$. For a detailed description of properties of the relative geometrical spreading $J(s_0, s)$, see [2]. In Eq. (4) we do not consider phase shifts due to caustics.

If the geometry of the rays is known, all quantities in Eq. (4) can be readily determined except for integration constants $C^{(K)}$ for $K > 0$. For point sources in homogeneous media, these constants are zero (see [14,15]), but in inhomogeneous media they are, in general, non-zero. The determination of the higher-order integration constants is probably the major problem in calculating the higher-order ray approximations. They should be determined so that the ray solution satisfies the initial and boundary conditions. The concrete way of determining the higher-order integration constants for $K > 0$ will be discussed later.

2.4. The ray amplitudes of the elastodynamic Green's function

The ray amplitudes of the elastodynamic Green's function, $U_{in}^{(K)}$, $K \geq 0$, can be obtained by calculating *additional and principal components* $U_{in}^{(K)\perp}$ and $U_{in}^{(K)||}$:

$$U_{in}^{W(K)} = U_{in}^{W(K)\perp} + U_{in}^{W(K)||},$$

where the superscript W denotes the P or S wave.

For P waves, we determine the additional and principal components in the following way [2]:

$$\text{for } K = 0: \; U_{in}^{P(0)\perp} = 0, \quad U_{in}^{P(0)||} = \frac{1}{4\pi\sqrt{\rho(s_0)\rho(s)v^P(s_0)v^P(s)}} \frac{g_i^P(s)g_n^P(s_0)}{J^P(s_0, s)},$$

$$\text{for } K > 0:$$

$$U_{in}^{P(K)\perp} = -\frac{\lambda(s)+2\mu(s)}{\rho(s)(\lambda(s)+\mu(s))}[M_{kn}(U_{in}^{P(K-1)}) - L_{kn}(U_{in}^{P(K-2)})](g_k^{S1}(s)g_i^{S1}(s) + g_k^{S2}(s)g_i^{S2}(s)),$$

$$U_{in}^{P(K)||} = \frac{g_i^P(s)}{2\sqrt{\rho(s)v^P(s)}J^P(s_0, s)} \left\{ C_n^{P(K)}(s_0, \gamma_1, \gamma_2) + \int_{s_0}^{s} \frac{J^P(s_0, s')}{\sqrt{\rho(s')\,v^P(s')}} \right.$$

$$\left. \times [L_{kn}(U_{kn}^{P(K-1)}) - M_{kn}(U_{in}^{P(K)\perp})]g_k^P(s')\,\mathrm{d}s' \right\}, \tag{6}$$

and for S waves we get

$$
\begin{aligned}
&\text{for } K=0: \; U_{in}^{S(0)\perp}=0, \quad U_{in}^{S(0)||}=\frac{1}{4\pi\sqrt{\rho(s_0)\rho(s)v^S(s_0)v^S(s)}}\frac{g_i^{S1}(s)g_n^{S1}(s_0)+g_i^{S2}(s)g_n^{S2}(s_0)}{J^S(s_0,s)},\\
&\text{for } K>0:\\
&U_{in}^{S(K)\perp}=\frac{\mu(s)}{\rho(s)(\lambda(s)+\mu(s))}[M_{kn}(U_{in}^{S(K-1)})-L_{kn}(U_{in}^{S(K-2)})]g_k^P(s)g_i^P(s),\\
&U_{in}^{S(K)||}=U_{in}^{S1(K)||}+U_{in}^{S2(K)||},\\
&U_{in}^{S1(K)||}=\frac{g_i^{S1}(s)}{2\sqrt{\rho(s)v^S(s)}J^S(s_0,s)}\left\{C_n^{S1(K)}(s_0,\gamma_1,\gamma_2)+\int_{s_0}^{s}\frac{J^S(s_0,s')}{\sqrt{\rho(s')v^S(s')}}\right.\\
&\qquad\left.\times[L_{kn}(U_{in}^{S(K-1)})-M_{kn}(U_{in}^{S(K)\perp})]g_k^{S1}(s')\,\mathrm{d}s'\right\},\\
&U_{in}^{S2(K)||}=\frac{g_i^{S2}(s)}{2\sqrt{\rho(s)v^S(s)}J^S(s_0,s)}\\
&\qquad\times\left\{C_n^{S2(K)}(s_0,\gamma_1,\gamma_2)+\int_{s_0}^{s}\frac{J^S(s_0,s')}{\sqrt{\rho(s')v^S(s')}}[L_{kn}(U_{in}^{S(K-1)})-M_{kn}(U_{in}^{S(K)\perp})]g_k^{S2}(s')\,\mathrm{d}s'\right\}.
\end{aligned}
\tag{7}
$$

Vectors $\mathbf{g}^P$, $\mathbf{g}^{S1}$ and $\mathbf{g}^{S2}$ are the basis vectors of the ray-centered coordinate system, v^P and v^S are the P- and S-wave velocities. Quantity $C_n^{W(K)}$, $W=P$, $S1$ and $S2$, denotes the integration constant for the Kth-order ray approximation of the P- and S-wave ray expansions, and quantity $J^W(s_0, s)$, $W=P$, S denotes the P and S relative geometrical spreadings from the source to the observation point. The geometrical spreadings can be expressed by formulae analogous to Eq. (5) for the acoustic case. Differential operators $L_{kn}(U_{in}^{(K-1)})$ and $M_{kn}(U_{in}^{(K)\perp})$ are defined as follows ([3], Eq. (2.11)):

$$
\begin{aligned}
M_{in}(U_{jn})&=(\lambda+\mu)[p_jU_{jn,i}+p_iU_{jn,j}+U_{jn}p_{i,j}]\\
&\quad+\mu[2p_jU_{in,j}+U_{in}p_{j,j}]+\lambda_{,i}U_{jn}p_j+\mu_{,j}[U_{in}p_j+U_{jn}p_i],\\
L_{in}(U_{jn})&=(\lambda+\mu)U_{jn,ij}+\mu U_{in,jj}+\lambda_{,i}U_{jn,j}+\mu_{,j}[U_{in,j}+U_{jn,i}],
\end{aligned}
\tag{8}
$$

where $\mathbf{p}$ is the slowness vector. We do not consider the phase shifts due to caustics in Eqs. (6) and (7). Note that $S1$ and $S2$ are not separate waves, but components of the S wave in the ray-centered coordinate system.

Similarly, to the acoustic case, the major problem in using Eqs. (6) and (7) is the determination of integration constants $C_n^{(K)}$, $K>0$. The way to calculate the integration constants is described in Section 4.

3. Isotropic medium with a weak velocity gradient

3.1. Definition

We shall consider an isotropic medium with constant density of the medium ρ and with weak and constant velocity gradient $\mathbf{b}$. Vector $\mathbf{b}$ can be expressed in terms of its magnitude b and unit vector $\mathbf{n}$: $\mathbf{b}=b\mathbf{n}$. For acoustic velocity $c(\mathbf{x})$, and for P- and S-wave velocities, $v^P(\mathbf{x})$ and $v^S(\mathbf{x})$, we obtain

$$c(\mathbf{x})=c_0(1-\mathbf{b}\cdot\mathbf{x}),\quad v^P(\mathbf{x})=v_0^P(1-\mathbf{b}\cdot\mathbf{x}),\quad v^S(\mathbf{x})=v_0^S(1-\mathbf{b}\cdot\mathbf{x}),\tag{9}$$

where c_0, v_0^P and v_0^S are the velocities at the source. We shall study waves only in the limited region for which

$$|\mathbf{b}\cdot\mathbf{x}|<br\ll 1,\quad \text{where } b=|\boldsymbol{b}| \text{ and } r=|\mathbf{x}|.\tag{10}$$

The region under study is a sphere around the source, in which velocities do not differ much from the velocity at the source. Therefore, we can consider the gradient medium as a perturbed medium of a homogeneous isotropic background. Hereafter, we shall consider only the first-order perturbation theory.

3.2. Analytical ray tracing

Condition (10) indicates that not only acoustic velocity $c(\mathbf{x})$, but also slowness $p(\mathbf{x})$ and the square of slowness $p^2(\mathbf{x})$ have approximately a constant and weak gradient:

$$p(\mathbf{x}) = p_0(1+\mathbf{b}\cdot\mathbf{x}), \quad p^2(\mathbf{x}) = p_0^2(1+2\mathbf{b}\cdot\mathbf{x}), \tag{11}$$

where p_0 is the slowness at the source, and we neglected the second-order perturbation $(\mathbf{b}\cdot\mathbf{x})^2$. We have neither mentioned equations for the elastodynamic case, because they are analogous to those in the acoustic case, nor shall we mention the equations for elastodynamics unless they differ from those for acoustics.

In the medium with a constant gradient of the square of slowness, the ray-tracing equations can be solved analytically. The rays, slowness vectors and traveltimes take the following simple form [2]:

$$\mathbf{x}(\sigma) = \mathbf{p}_0\sigma + \tfrac{1}{2}\mathbf{b}p_0^2\sigma^2, \quad \mathbf{p}(\sigma) = \mathbf{p}_0 + \mathbf{b}p_0^2\sigma, \quad \tau(\sigma) = p_0^2\sigma + \mathbf{b}\cdot\mathbf{p}_0 p_0^2\sigma^2 + \tfrac{1}{3}b^2p_0^4\sigma^3, \tag{12}$$

where σ is the parameter along the ray. The coordinate system in Eq. (12) has its origin at the source, and the traveltime at the source is zero. Vector $\mathbf{p}_0 = p_0(\sin\theta_0\cos\varphi_0,\ \sin\theta_0\sin\varphi_0,\ \cos\theta_0)^{\mathrm{T}}$ is the slowness vector at the source defined by its magnitude $p_0 = 1/c_0$ and by take-off angles θ_0 and φ_0. It follows from Eq. (12) that the ray field is axially symmetric around vector $\mathbf{b}$. Traveltime τ in Eq. (12) and the arclength of ray s can be approximately related to parameter σ by using approximation (10) as follows:

$$\tau(\sigma) = p_0^2\sigma(1+\mathbf{b}\cdot\mathbf{p}_0\sigma), \quad s(\sigma) = p_0\sigma\,(1+\tfrac{1}{2}\mathbf{b}\cdot\mathbf{p}_0\sigma). \tag{13}$$

Parameter σ and arclength s can also be related to the distance from the observation point to the source:

$$\sigma = rc_0(1-\tfrac{1}{2}\mathbf{b}\cdot\mathbf{x}), \quad s = r. \tag{14}$$

Although, the rays are not straight lines in the first-order perturbation theory, the length of the ray is approximately equal to the distance from the observation point to the source.

3.3. Relative geometrical spreading and polarization vectors

By differentiating Eq. (12) we can derive simple analytical formulae for determinants $\det\mathbf{Q}(s)$ and $\det\mathbf{P}(s_0)$, and for relative geometrical spreading $J(s_0, s)$ in Eq. (5):

$$\det\mathbf{Q}(s) = p_0^2\sigma^2\sin\theta_0, \quad \det\mathbf{P}(s_0) = p_0^2\sin\theta_0, \quad J(s_0,s) = \sigma, \qquad 15$$

where θ_0 is the angle between the slowness vector at the source and vector $\mathbf{b}$. Parameter σ is related to distance r by formula (14).

In contrast to the acoustic equation (4), for the elastodynamic equations (6) and (7) we also need to specify polarization vectors $\mathbf{g}^P$, $\mathbf{g}^{S1}$ and $\mathbf{g}^{S2}$ which are mutually perpendicular. Vector $\mathbf{g}^P$ is parallel to the tangent to the ray, $\mathbf{g}^{S1}$ lies in the plane defined by vectors $\mathbf{b}$ and $\mathbf{g}^P$, and vector $\mathbf{g}^{S2}$ is perpendicular to this plane. If vector $\mathbf{b}$ is vertical, then the polarization vectors $\mathbf{g}^{S1}$and $\mathbf{g}^{S2}$ become $\mathbf{g}^{SV}$ and $\mathbf{g}^{SH}$, respectively. Vector $\mathbf{g}^P$ is approximately expressed as follows:

$$\mathbf{g}^P = (\mathbf{g}_0^P + \mathbf{b}p_0\sigma)(1-\mathbf{b}\cdot\mathbf{p}_0\sigma), \tag{16}$$

where $\mathbf{g}_0^P$ specifies the direction of the slowness vector (or the direction of the ray) at the source.

4. Higher-order ray approximations

Using Eqs. (4), (6) and (7), we can calculate higher-order ray amplitudes $U^{(K)}(\mathbf{x})$, $U_{kl}^{P(K)}(\mathbf{x})$ and $U_{kl}^{S(K)}(\mathbf{x})$. Subsequently, we can construct the complete ray expansions of the acoustic and elastodynamic Green's functions. Since we are dealing with a very simple type of medium and we linearize all formulae using the first-order perturbation theory, we can calculate the higher-order ray approximations analytically to an arbitrary order. However, since the formulae for the higher-order ray approximations are recursive, some extensive algebra is required to deal with the rather complex formulae. We, therefore, performed these calculations using symbolic manipulation software REDUCE [6]. In this section, we shall not present a detailed derivation, but shall only summarize some basic formulae and general results. The final formulae for the acoustic and elastodynamic Green's functions will be given in Section 5.

By calculating the higher-order ray approximations of the acoustic and elastodynamic Green's functions we arrive at the following results:

1. *The ray series for the acoustic as well as elastodynamic Green's functions have a finite number of terms*. Therefore, we can construct the complete ray-theory formulae for the Green's functions, correct not only for high frequencies such as the zeroth-order ray approximation, but in the whole frequency range.
2. Calculating the ray expansion of the acoustic Green's function, we find that the only non-zero term of the ray series is the zeroth-order term. All higher-order terms are equal to zero. Also higher-order integration constants $C^{(K)}$, $K>0$, are equal to zero because the zeroth-order Green's function satisfies the initial conditions at the source and radiation conditions in infinity.
3. If we neglect the higher-order integration constants in calculating the elastodynamic Green's function, we find that four higher-order ray approximations are non-zero. This applies to P waves as well as to S waves. All the others terms of the ray series are zero. The sum of the expansions of the P- and S-wave Green's functions, however, does not meet the radiation conditions because it diverges at any observation point $\mathbf{x}$ with increasing time t. Such behaviour is obviously unphysical. In homogeneous media, the P- and S-wave ray expansions of the Green's function are also divergent in time, but their sum leads to the cancellation of this divergency [15]. In the weakly inhomogeneous medium studied, however, the sum of the P- and S-wave ray expansions does not eliminate the divergency. Therefore, we conclude that we must include *non-zero higher-order integration constants* into the ray series to obtain a correct solution.
4. As mentioned in the previous sections, a general way of determining the higher-order integration constants of the ray-theory Green's function is not known. Fortunately, the assumption of the weak velocity gradient in the medium simplifies the problem considerably. Since the ray series under study is finite comprising only five terms including the zeroth-order term, we have to determine only a few higher-order integration constants. Moreover, we know that the integration constants must be of the order of the first perturbation because the constants are zero in the homogeneous background medium [15]. Stipulating that the elastodynamic Green's function does not diverge in time, we can determine the following integration constants:

$$C_k^{P(1)} = \frac{\mathbf{b}\cdot\mathbf{x}}{4\pi\sqrt{\rho\alpha}r}\,\frac{\delta_{k3}(\kappa+7)-g_k^P(5\kappa+3)}{\kappa-1},$$
$$C_k^{S1(1)} = -\frac{\mathbf{b}\cdot\mathbf{x}}{4\pi\sqrt{\rho\beta}r}\left[\frac{\tan^2\theta_0(\kappa+7)}{\kappa-1}+2\right]g_k^{S1},\quad C_k^{S2(1)} = -\frac{\mathbf{b}\cdot\mathbf{x}}{2\pi\sqrt{\rho\beta}r}\,g_k^{S2}, \tag{17}$$

   where α and β are the P and S velocities at the source, $\kappa=\alpha^2/\beta^2$ is the square of the P-to-S velocity ratio, and θ_0 is the angle between the slowness vector $\mathbf{p}_0$ at the source and the velocity gradient vector $\mathbf{b}$. All the other higher-order integration constants are zero.
5. If we include integration constants (17) in the ray solution, the number of non-zero higher-order ray approximations reduces from four to two. Thus the *elastodynamic Green's function in inhomogeneous isotropic media with the weak velocity gradient is expressed by the same number of higher-order ray approximations as the Green's function in homogeneous isotropic media*.

5. Final formulae for the Green's functions

5.1. Acoustic Green's function

The final form of the acoustic Green's function $G(\mathbf{x}, t)$ in an unbounded isotropic medium with constant density, and with a weak and constant velocity gradient reads

$$G(\mathbf{x}, t) = \frac{\rho}{4\pi r}\delta\,(t-\tau)\,, \quad \text{where } \tau = \frac{r}{c_0}(1 + \tfrac{1}{2}\mathbf{b}\cdot\mathbf{x}), \quad |\mathbf{b}\cdot\mathbf{x}| < |\mathbf{b}|\,|\mathbf{x}| \ll 1. \tag{18}$$

Vector $\mathbf{b}$ denotes the weak velocity gradient, $\mathbf{x}$ is the position vector of the observation point, r the distance from the observation point to the source, and c_0 is the acoustic velocity at the source. Interestingly, *Green's function* (18) *is expressed by the zeroth-order ray approximation only and has the same form as in homogeneous media,* only traveltime τ is different. Our result can also be compared with the explicit closed-form solution for the constant gradient of the square of slowness found by Li et al. [11]. The mentioned authors expressed the acoustic Green's function in terms of Airy functions for media even with a strong gradient of the square of slowness. *By linearizing the solution by Li et al.*[11] *using condition* (10) *we obtain exactly the ray-theory solution* (18). This proves that the ray-theory approach is correct.

Physically, the acoustic Green's function represents the acoustic pressure generated by a unit volume source. Formula (18) for the pressure can be transformed into the formula for the displacement as follows:

$$u_k(\mathbf{x}, t) = -\frac{1}{\rho^2 c_0^2} G_{,k}\,(\mathbf{x}, t)\,. \tag{19}$$

The normalization factor in Eq. (19) has been chosen to render the acoustic solution directly comparable with the elastodynamic solution generated by the same kind of source. Eqs. (18) and (19) yield

$$\begin{aligned} &u_k(\mathbf{x}, t) = u_k^0(\mathbf{x}, t) + b u_k^{\text{pert}}(\mathbf{x}, t), \\ &u_k^0(\mathbf{x}, t) = \frac{1}{4\pi\rho}\left\{\frac{N_k}{c_0^3 r}\dot{\delta}(t-\tau) + \frac{N_k}{c_0^2 r^2}\delta(t-\tau)\right\}, \quad u_k^{\text{pert}}(\mathbf{x}, t) = \frac{1}{16\pi\rho}\frac{N_k N_l n_l + n_k}{c_0^3}\dot{\delta}(t-\tau), \end{aligned} \tag{20}$$

where $u_k^0(\mathbf{x}, t)$ is the solution in the background homogeneous medium, and $u_k^{\text{pert}}(\mathbf{x}, t)$ is its perturbation. The displacement solution (20) is expressed by the zeroth- and first-order ray approximations and differs from the solution in homogeneous media with the non-zero perturbation part $u_k^{\text{pert}}(\mathbf{x}, t)$.

5.2. Elastodynamic Green's function

The final form of the elastodynamic Green's function $G_{kl}(\mathbf{x}, t)$ can be expressed as the sum of the Green's function in the background homogeneous medium $G_{kl}^0(\mathbf{x}, t)$ and its perturbation $G_{kl}^{\text{pert}}(\mathbf{x}, t)$:

$$\begin{aligned} G_{kl}(\mathbf{x}, t) &= G_{kl}^0(\mathbf{x}, t) + b G_{kl}^{\text{pert}}(\mathbf{x}, t), \\ G_{kl}^0(\mathbf{x}, t) &= \frac{1}{4\pi\rho}\left\{\frac{N_k N_l}{\alpha^2}\frac{1}{r}\delta(t-\tau^P) + \frac{\delta_{kl} - N_k N_l}{\beta^2}\frac{1}{r}\delta(t-\tau^S) + (3N_k N_l - \delta_{kl})\frac{1}{r^3}\int_{\tau^P}^{\tau^S}\tau\delta(t-\tau)\,\mathrm{d}\tau\right\}, \\ G_{kl}^{\text{pert}}(\mathbf{x}, t) &= \frac{1}{4\pi\rho}\left\{\frac{1}{\alpha^2}(N_k N_l N_j n_j + \frac{1}{2}N_l n_k - \frac{1}{2}N_k n_l)\delta(t-\tau^P) + \frac{1}{\beta^2}\left(-N_k N_l N_j n_j + \frac{1}{2}N_l n_k\right.\right. \\ &\quad \left.\left. - \frac{1}{2}N_k n_l + N_j n_j \delta_{kl}\right)\delta(t-\tau^S) + \frac{4}{\kappa - 1}(-N_l n_k + N_k n_l)\frac{1}{r^2}\int_{\tau^P}^{\tau^S}\tau\delta(t-\tau)\,\mathrm{d}\tau\right\}, \end{aligned} \tag{21}$$

where b is the magnitude of velocity gradient $\mathbf{b}$, unit vector $\mathbf{n}$ specifies the direction of $\mathbf{b}$, α and β are the P and S velocities at the source, $\kappa=\alpha^2/\beta^2$ is the square of the P-to-S velocity ratio which is constant in the medium, ρ the density of the medium which is constant, r the distance of the observation point from the source, $\mathbf{N}=\mathbf{x}/r$ is the unit direction vector pointing to the observation point, τ^P and τ^S are the P- and S-wave traveltimes:

$$\tau^P=\frac{r}{\alpha}(1+\tfrac{1}{2}\mathbf{b}\cdot\mathbf{x}),\quad \tau^S=\frac{r}{\beta}(1+\tfrac{1}{2}\mathbf{b}\cdot\mathbf{x}),\ \text{where } |\mathbf{b}\cdot\mathbf{x}|<br\ll 1. \tag{22}$$

The elastodynamic Green's function (21) consists of three ray approximations. The correctness of Eq. (21) was verified by its inserting into elastodynamic equation (1b).

From Eq. (21) we can derive wavefield $u_k(\mathbf{x},t)=G_{kl,l}(\mathbf{x},t)$ generated by an explosive source:

$$\begin{aligned}
u_k(\mathbf{x},t)&=u_k^0(\mathbf{x},t)+bu_k^{\text{pert}}(\mathbf{x},t),\\
u_k^0(\mathbf{x},t)&=\frac{1}{4\pi\rho}\left\{\frac{N_k}{\alpha^3 r}\dot\delta(t-\tau^P)+\frac{N_k}{\alpha^2 r^2}\delta(t-\tau^P)\right\},\ u_k^{\text{pert}}(\mathbf{x},t)\\
&=\frac{1}{8\pi\rho}\left\{\frac{3N_kN_ln_l+n_k}{\alpha^3}\ \dot\delta(t-\tau^P)-2\frac{N_kN_ln_l-n_k}{\beta^3}\dot\delta(t-\tau^P)+2\frac{\kappa+3}{\kappa-1}\frac{N_kN_ln_l-n_k}{\alpha^2 r}\delta(t-\tau^P)\right.\\
&\quad\left.-2\frac{(\kappa+3)N_kN_ln_l+(\kappa-5)n_k}{\beta^2(\kappa-1)r}\delta(t-\tau^S)+8\frac{3N_kN_ln_l-n_k}{(\kappa-1)r^3}\int_{\tau^P}^{\tau^S}\tau\delta(t-\tau^S)\,\mathrm{d}\tau\right\}.
\end{aligned}\tag{23}$$

As expected, the explosive source in the gradient medium generates not only the P wave but also the S wave. The amplitude of the S wave is small as compared to the amplitude of the P wave, and is of the order of the perturbation. Moreover, the radiation function of the P wave is not spherically symmetric as in homogeneous media, but it is more complicated. The deviation from the spherically symmetric radiation function is again of the order of the perturbation. Formula (23) consists of four ray approximations, while the analogous formula for the explosive source in homogeneous media consists only of two ray approximations.

Finally, we should note that formula (23) is essentially different from formula (20) for the displacement generated by an explosive source in the acoustic medium. The explosive source generates only P waves in the acoustic medium, but it also generates S waves in the elastodynamic medium. This difference is caused by the different models used. Although both the models display the same P-wave velocity gradient, they differ in the distribution of the velocity of S waves. The acoustic medium is defined by zero S-wave velocity with no velocity gradient, but for the elastodynamic medium we assumed a non-zero S-wave velocity with non-zero velocity gradient.

5.3. Elastostatic Green's function

The elastostatic Green's function $G_{kl}(\mathbf{x})$ can be obtained from the elastodynamic Green's function $G_{kl}(\mathbf{x},t)$ by convolution with the unit constant in time. Analogously, to the elastodynamic Green's function we can divide the elastostatic Green's function into the Green's function in the background homogeneous medium $G_{kl}^0(\mathbf{x})$ and into its perturbation $G_{kl}^{\text{pert}}(\mathbf{x})$:

$$\begin{aligned}
G_{kl}(\mathbf{x})&=G_{kl}^0(\mathbf{x})+bG_{kl}^{\text{pert}}(\mathbf{x}),\\
G_{kl}^0(\mathbf{x})&=\frac{1}{8\pi\rho\alpha^2}\frac{1}{r}\left\{N_kN_l(\kappa-1)+\delta_{kl}(\kappa+1)\right\},\ G_{kl}^{\text{pert}}(\mathbf{x})\\
&=\frac{1}{8\pi\rho\alpha^2}\left\{(\kappa-1)N_kN_lN_jn_j+(\kappa+1)N_jn_j\delta_{kl}+(\kappa-3)N_ln_k-(\kappa-3)N_kn_l\right\}.
\end{aligned}\tag{24}$$

Proof of the elastostatic Green's tensor being correct was obtained by inserting Eq. (24) into the elastostatic equation.

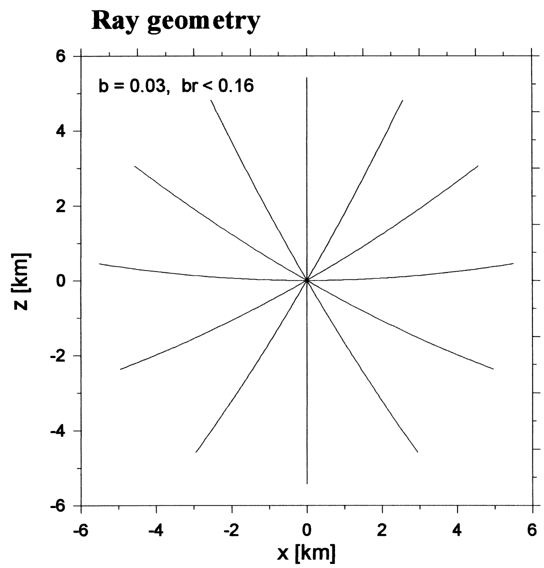

Fig. 3. Ray geometry for a point source situated in a medium with a weak velocity gradient.

6. Numerical example

We shall now study the properties of the elastodynamic Green's function (21) numerically. We shall not show properties of the acoustic Green's function (18) because its form is very simple and its properties are obvious. As the model of the medium, we shall use a vertically inhomogeneous medium in which the *P*- and *S*-wave velocities

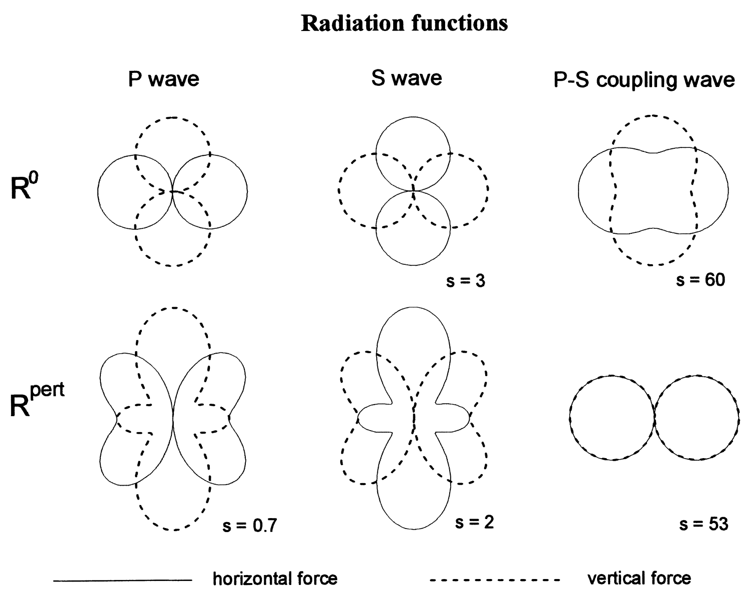

Fig. 4. Radiation functions for *P*, *S* and *P–S* coupling waves shown in the *x–z* plane. The radiation functions are defined as the magnitude of the amplitude of waves at distance $r = 1$ km from the source. Quantity *s* denotes the relative scale with respect to the radiation function of the *P* wave.

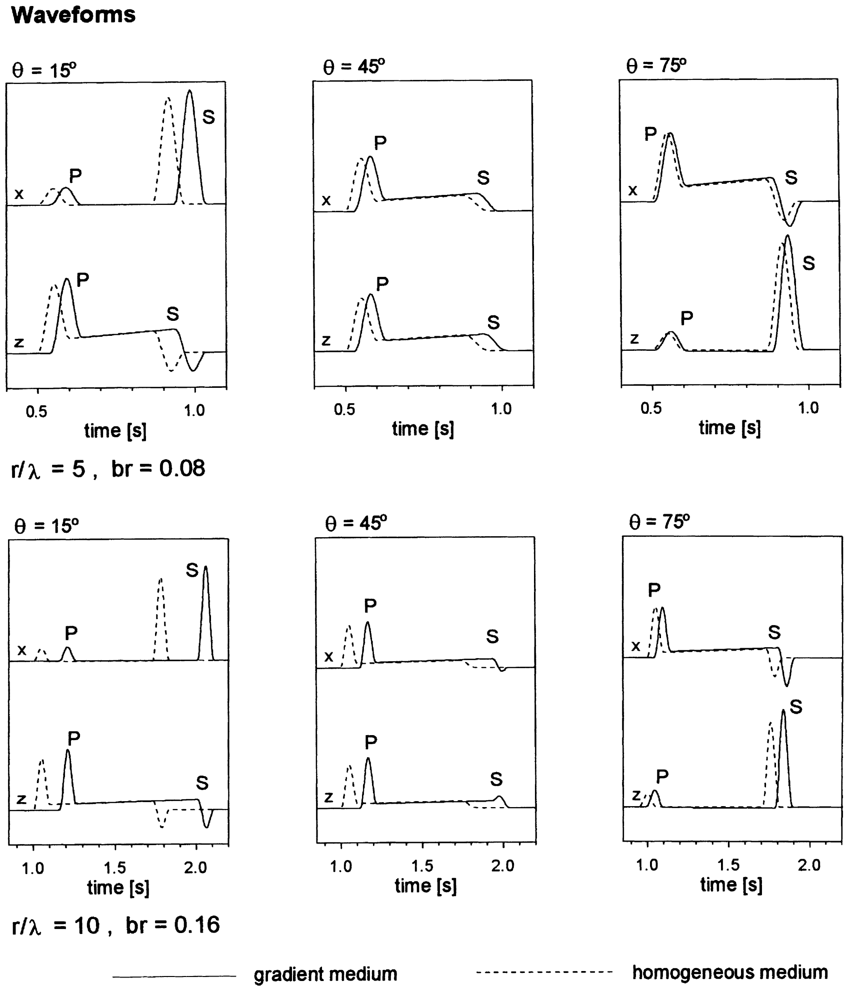

Fig. 5. Comparison of waveforms in the gradient medium and homogeneous background medium. θ is the angle between the position vector and the vertical, $\lambda=\alpha T$ is the wavelength of the *P* wave at the source, *r* is the distance of the observation point from the source, and *br* denotes the quantity from Eq. (10).

at the source are $\alpha=5.5$ km/s and $\beta=\alpha/\sqrt{3}$, respectively, and the density of the medium is $\rho=2.9$ g/cm^3. The velocity gradient is $b=0.03$. The density of the medium is constant. The ray geometry for a point source situated in such a medium is shown in Fig. 3. For calculating the wavefield in the medium, we use the single point force $\mathbf{F}(t)=\mathbf{F}f(t)$, where $\mathbf{F}$ is the force vector, and $f(t)$ is the source time function defined as follows:

$$f(t)=\sin^2\left(\frac{\pi t}{T}\right) \quad \text{for } t\in\langle 0,T\rangle,$$

for other times $f(t)$ is zero. The source time function has been chosen so that the wave has the form of a one-sided pulse in the far-field approximation for waves propagating in the homogeneous background medium. The pulse width is assumed to be $T=0.1$ s. We choose force vectors $\mathbf{F}=(1,0,0)^{\mathrm{T}}$ and $\mathbf{F}=(0,0,1)^{\mathrm{T}}$ in calculating the radiation functions, and $\mathbf{F}=(1,0,1)^{\mathrm{T}}$ in calculating the waveforms and particle motions.

The elastodynamic Green's function (21) consists of three waves: the *P* wave, *S* wave and *P–S* coupling wave. These three waves are present in both parts of the Green's function: in the Green's function for the homogeneous

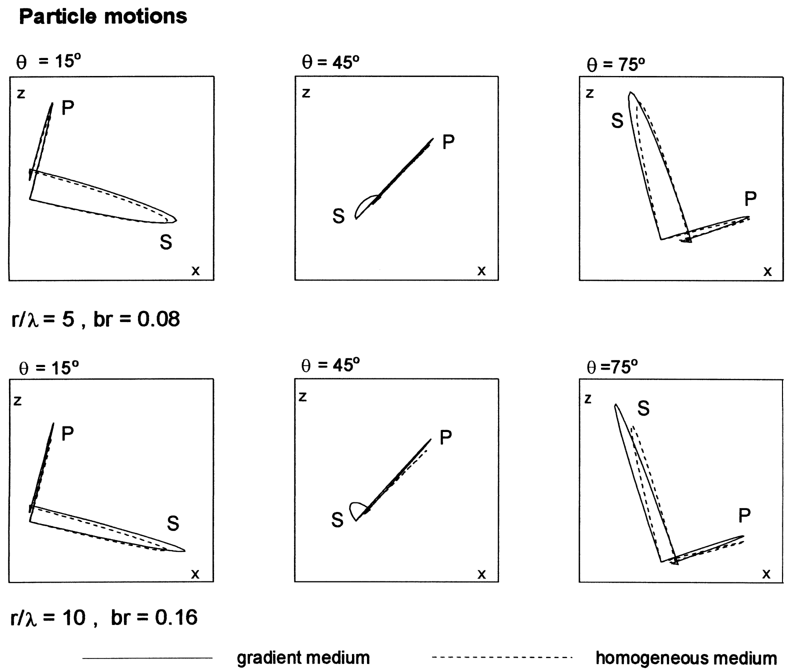

Fig. 6. Particle motions of waves in the gradient medium and homogeneous background medium.

background medium $G^0_{kl}(\mathbf{x}, t)$ as well as in its perturbation $G^{\mathrm{pert}}_{kl}(\mathbf{x}, t)$. In the background medium, the amplitude of the *P* and *S* wave decreases with distance as $1/r$, the amplitude of the *P–S* coupling wave as $1/r^3$. The amplitude of the *P*-wave and *S*-wave perturbations does not depend on the distance at all, and the amplitude of the perturbation of the *P–S* coupling wave decreases as $1/r^2$. A directional dependence of the amplitudes at a fixed distance from the source is displayed by radiation functions. Fig. 4 shows the radiation functions of the elastodynamic wavefield generated by horizontal and vertical point forces. The radiation functions are shown in the *x–z* plane separately for the *P* wave, *S* wave and for the *P–S* coupling wave. Moreover, they are divided into two parts: the radiation function in the homogeneous background medium and its perturbation due to the velocity gradient. Fig. 5 shows the waveforms of a complete wavefield in the homogeneous and gradient media at six observation points. The observation points lie in the *x–z* plane at distances of 5 and 10 wavelengths from the source: $r = 2.75$ and 5.50 km, and are specified by angle θ between the position vector and the vertical: $\theta = 15^\circ$, 45° and 75°. Major differences between the Green's function in the homogeneous medium and in the gradient medium are caused by differences in traveltimes, minor differences can be observed in the amplitudes of the waves. Both differences are directionally dependent. This can also be seen from the particle motions (see Fig. 6). The particle motions indicate that not only the amplitude but also the polarization direction of waves can differ in homogeneous and gradient media. This difference is particularly remarkable for directions nearly perpendicular to the velocity gradient vector. In Fig. 6 we can also observe the ellipticity of *P* and *S* waves caused by the *P–S* coupling wave. For more distant observation points, the ellipticity decreases and the *P–S* coupling wave vanishes.

7. Conclusion

We have derived explicit approximate analytical formulae for the acoustic, elastodynamic and elastostatic Green's functions in homogeneous isotropic media with a weak and constant velocity gradient. The formulae were obtained by *calculating the higher-order approximations of the ray series in the asymptotic ray theory analytically*. Since a weak velocity gradient is assumed, we were able to apply the first-order perturbation theory. Under this assumption, the calculation of the higher-order ray approximations simplified considerably. The *major problem in calculating the ray-theory Green's functions was to determine the higher-order integration constants* necessary for the proper determination of the higher-order ray approximations. In contrast to homogeneous media where the integration constants are zero, in inhomogeneous media the constants are, in general, non-zero. We managed to determine the integration constants by applying the condition that the complete Green's function should not be divergent in time with any position vector $\mathbf{x}$.

We found that the whole ray series of the acoustic Green's function consists of only one non-zero term which is the zeroth-order term. All higher-order approximations are zero. Therefore, the form of the acoustic Green's function is extremely simple and does not differ from the Green's function in homogeneous media: only corrections for different traveltimes must be considered. The whole ray series of the elastodynamic Green's function consists of three non-zero terms including the zeroth-order term. Since the ray series for the acoustic as well as elastodynamic Green's functions is finite, the *ray solution is complete and correct in the whole frequency range*. This is of particular interest because the ray method is usually assumed to be only a high-frequency approximation. The obtained formulae are applicable only to media with a weak velocity gradient and to the region around the source, where the following assumption is valid: $br \ll 1$, where b is the magnitude of the velocity gradient and r is the distance of the observation point from the source.

Interestingly, the ray series of the acoustic as well as elastodynamic *Green's functions in media with a weak gradient have the same number of terms as in homogeneous media*. In media with a stronger gradient, the higher-order perturbation theory should be applied, and the ray series will probably consist of a higher number of terms. The numerical comparison of the elastodynamic Green's functions in homogeneous and gradient media shows that the differences in the waveforms are mainly caused by different arrival times of the waves, the differences in the amplitudes are not as pronounced. Also the particle motions of waves in both the types of media are very similar, only minor differences in the prevailing polarization directions are observed.

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