Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversely isotropic

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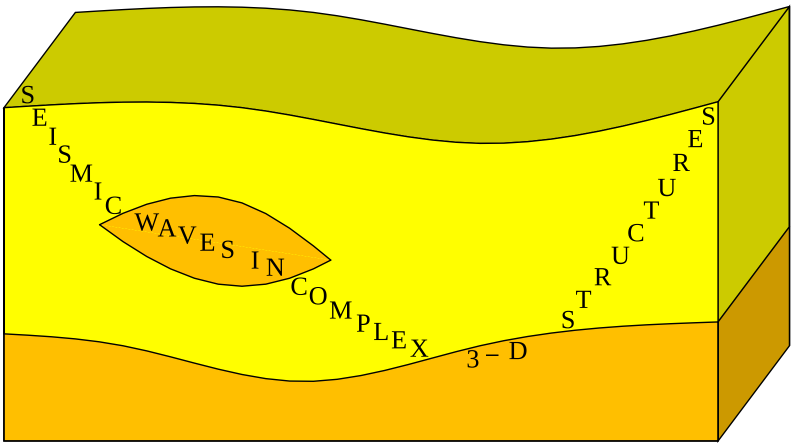

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For a given stiffness tensor (tensor of elastic moduli) a_{ijkl} of a generally anisotropic medium, we estimate to which extent is the medium transversely isotropic and determine the direction of its reference symmetry axis in terms of the reference symmetry vector.

We consider the rotation $a_{ijkl}(\varphi)$ of stiffness tensor $a_{ijkl} = a_{ijkl}(0)$ about given unit vector t_i by angle φ. The derivative of the stiffness tensor with respect to the angle φ of rotation is

$$a'_{ijkl} = \frac{\mathrm{d}a_{ijkl}}{\mathrm{d}\varphi}(0) \quad .$$

We choose the square

$$y = a'_{ijkl}\, a'_{ijkl} \tag{6}$$

of the norm of the derivative of the stiffness tensor with respect to the angle of rotation as the objective function.

The objective function can be expressed as the quadratic form

$$y = t_m B_{mn} t_n \tag{7}$$

with positive-semidefinite 3×3 matrix B_{mn}. We denote the eigenvalues of matrix B_{mn} from the greatest to the smallest by $B_{(1)}$, $B_{(2)}$ and $B_{(3)}$, and the corresponding unit eigenvectors by $t_{i(1)}$, $t_{i(2)}$ and $t_{i(3)}$.

Ratio

$$\rho = \sqrt{\frac{a'_{ijkl}\, a'_{ijkl}}{a_{ijkl}\, a_{ijkl}}} \tag{9}$$

characterizes the extent of the dependence of the stiffness tensor on the rotation.

For the reference symmetry vector $t_{i(3)}$, ratio (9) reads

$$\rho_{(3)} = \sqrt{\frac{B_{(3)}}{a_{ijkl}\, a_{ijkl}}} \quad . \tag{10}$$

This ratio characterizes the extent to which the medium is not transversely isotropic. We shall thus refer to it as the non-TI ratio.

The reference symmetry vector $t_{i(3)}$ is stable and has a good physical meaning only if the minimum eigenvalue $B_{(3)}$ of matrix B_{mn} is considerably smaller than other two eigenvalues $B_{(1)}$ and $B_{(2)}$, i.e., if $\rho_{(3)}$ is considerably smaller than ratios

$$\rho_{(A)} = \sqrt{\frac{B_{(A)}}{a_{ijkl}\, a_{ijkl}}} \quad , \qquad A = 1, 2 \quad . \tag{11}$$

If non-TI ratio (10) is zero within rounding errors, the medium is transversely isotropic and reference symmetry vector $t_{i(3)}$ specifies its symmetry axis.

We can also calculate the first-order and second-order spatial derivatives of the reference symmetry vector (Klimeš, 2015, secs. 2.3, 2.4).

These derivatives may be useful for tracing the SH and SV reference rays (Klimeš & Bulant, 2014; 2015) in heterogeneous velocity models with spatially varying reference symmetry vector, and for solving the corresponding equations of geodesic deviation (dynamic ray tracing).

Numerical examples

Unit reference symmetry vector $t_{i(3)}$ and non-TI ratio $\rho_{(3)}$ for a given stiffness tensor a_{ijkl} is determined by new program `tiaxis.for` of software package FORMS (Bucha & Bulant, 2015).

Velocity model WA at the surface

The surface value of the density reduced stiffness tensor in 1-D anisotropic velocity model WA in $\mathrm{km}^2\mathrm{s}^{-2}$ (Pšenčík & Dellinger, 2001):

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\ & 15.71 & 5.04 & 0.00 & 0.00 & 0.00 \\ & & 15.71 & 0.00 & 0.00 & 0.00 \\ & & & 5.33 & 0.00 & 0.00 \\ & & & & 4.98 & 0.00 \\ & & & & & 4.98 \end{pmatrix} \end{array} \quad . \tag{20}$$

Non-TI ratio:

$$\rho_{(3)} = 0.000847 \quad . \tag{21}$$

Unit reference symmetry vector:

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \tag{22}$$

The medium is not exactly transversely isotropic but is approximately transversely isotropic.

We now slightly change the surface value of the density reduced stiffness tensor (20) of velocity model WA to density reduced stiffness tensor (in $\mathrm{km}^2\mathrm{s}^{-2}$)

$$
\begin{array}{c}
\\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
13.39 & 4.46 & 4.46 & 0.00 & 0.00 & 0.00 \\
 & 15.70 & 5.04 & 0.00 & 0.00 & 0.00 \\
 & & 15.70 & 0.00 & 0.00 & 0.00 \\
 & & & 5.33 & 0.00 & 0.00 \\
 & & & & 4.98 & 0.00 \\
 & & & & & 4.98
\end{pmatrix}
\end{array}
\qquad (23)
$$

of a transversely isotropic medium.

Non-TI ratio:

$$\rho_{(3)} = 0.000000 \quad . \qquad (24)$$

Unit reference symmetry vector:

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \qquad (25)$$

Velocity model QI at the surface

The stiffness tensor of velocity model WA was rotated by 45° about the positive x_3 half-axis in order to create vertically heterogeneous 1-D anisotropic velocity model QI. The density reduced stiffness tensor in $\mathrm{km}^2\mathrm{s}^{-2}$ (Bulant & Klimeš, 2002, eq. 38; 2008, eq. 12; Klimeš & Bulant, 2004, eq. 57; Pšenčík, Farra & Tessmer, 2012, eq. 16):

$$
\begin{array}{c}
\\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix}
14.485 & 4.525 & 4.750 & 0.000 & 0.000 & -0.580 \\
 & 14.485 & 4.750 & 0.000 & 0.000 & -0.580 \\
 & & 15.710 & 0.000 & 0.000 & -0.290 \\
 & & & 5.155 & -0.175 & 0.000 \\
 & & & & 5.155 & 0.000 \\
 & & & & & 5.045
\end{pmatrix}
\end{array} \quad . \qquad (26)
$$

Non-TI ratio:

$$\rho_{(3)} = 0.000847 \quad . \qquad (27)$$

Unit reference symmetry vector:

$$t_{i(3)} = (0.707107 \;\; 0.707107 \;\; 0.000000) \quad . \qquad (28)$$

The medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

Velocity model KISS at the surface

The stiffness tensor of velocity model WA was rotated by 1° about the positive x_3 half-axis in order to create vertically heterogeneous 1-D anisotropic velocity model KISS. The density reduced stiffness tensor in km^{2}s^{-2} (Pšenčík, Farra & Tessmer, 2012, eq. 20):

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\
\begin{pmatrix} 13.39063 & 4.46008 & 4.46018 & 0.00000 & 0.00000 & -.01797 \\ & 15.70921 & 5.03982 & 0.00000 & 0.00000 & -.02251 \\ & & 15.71000 & 0.00000 & 0.00000 & -.01012 \\ & & & 5.32989 & -.00611 & 0.00000 \\ & & & & 4.98011 & 0.00000 \\ & & & & & 4.98008 \end{pmatrix} \end{array} . \qquad (29)$$

Non-TI ratio:

$$\rho_{(3)} = 0.000848 \quad . \qquad (30)$$

Unit reference symmetry vector:

$$t_{i(3)} = (0.999848 \;\; 0.017452 \;\; 0.000000) \quad . \qquad (31)$$

The medium is not exactly transversely isotropic but is approximately transversely isotropic, analogously to velocity model WA.

Velocity model SC1

The density reduced stiffness tensor in homogeneous anisotropic velocity model 1 by Shearer & Chapman (1989) in $\mathrm{km}^2\mathrm{s}^{-2}$:

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 20.04 & 7.41 & 7.41 & 0.00 & 0.00 & 0.00 \\ & 20.22 & 7.46 & 0.00 & 0.00 & 0.00 \\ & & 20.22 & 0.00 & 0.00 & 0.00 \\ & & & 6.38 & 0.00 & 0.00 \\ & & & & 5.10 & 0.00 \\ & & & & & 5.10 \end{pmatrix} \end{array} \quad . \tag{32}$$

Non-TI ratio:

$$\rho_{(3)} = 0.000000 \quad . \tag{33}$$

Unit reference symmetry vector:

$$t_{i(3)} = (1.000000 \;\; 0.000000 \;\; 0.000000) \quad . \tag{34}$$

The medium is transversely isotropic within the rounding errors. If we inspect manually stiffness tensor (32), we see that the medium is exactly transversely isotropic.

Velocity model SC1_II at the surface

The stiffness tensor of velocity model 1 by Shearer & Chapman (1989) was first rotated by 45° about the positive x_2 half-axis and then rotated by 30° about the positive x_3 half-axis. The density reduced stiffness tensor in km^{2}s^{-2} (Pšenčík, Farra & Tessmer, 2012, eq. 19):

$$
\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad 11 \quad & \quad 22 \quad & \quad 33 \quad & \quad 23 \quad & \quad 13 \quad & \quad 12 \quad \end{array} \\
\left(\begin{array}{rrrrrr}
18.97125 & 7.67125 & 8.36125 & 0.46000 & -0.31177 & -0.15589 \\
 & 19.64625 & 7.74375 & -0.49500 & 0.25115 & -0.42868 \\
 & & 18.87000 & -0.02250 & -0.03897 & 0.53477 \\
 & & & 5.89500 & 0.26847 & -0.28146 \\
 & & & & 6.20500 & 0.15250 \\
 & & & & & 5.97625
\end{array}\right)
\end{array} . \tag{36}
$$

Non-TI ratio:

$$\rho_{(3)} = 0.000054 \quad . \tag{37}$$

Unit reference symmetry vector:

$$t_{i(3)} = (0.612372 \;\; 0.353554 \;\; 0.707107) \quad . \tag{38}$$

The medium is not exactly transversely isotropic but is close to transversely isotropic. Numerically determined unit reference symmetry vector (38) corresponds to the above described rotations.

Velocity model SC1_II at the depth of 1.4 km

Slightly cubic density reduced stiffness tensor in velocity model SC1_II at the depth of 1.5 km in km^2s^{-2} (Pšenčík, Farra & Tessmer, 2012, eq. 19):

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array}\begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 30.25 & 10.08 & 10.08 & 0.00 & 0.00 & 0.00 \\ & 30.25 & 10.08 & 0.00 & 0.00 & 0.00 \\ & & 30.25 & 0.00 & 0.00 & 0.00 \\ & & & 10.08 & 0.00 & 0.00 \\ & & & & 10.08 & 0.00 \\ & & & & & 10.08 \end{pmatrix} \end{array} \quad . \tag{39}$$

The elements of the stiffness tensor are linear functions of depth. At depths between 0 km and 1.5 km, velocity model SC1_II is close to transversely isotropic, but is slightly tetragonal. Non-TI ratio at the depth of 1.4 km:

$$\rho_{(3)} = 0.000397 \quad . \tag{40}$$

Unit reference symmetry vector at the depth of 1.4 km:

$$t_{i(3)} = (0.611611 \ \ 0.348810 \ \ 0.710115) \quad . \tag{41}$$

The velocity model is less transversely isotropic at the depth of 1.4 km than at the surface.

References:

Bucha, V. & Bulant, P. (eds.) (2015): SW3D-CD-19 (DVD-ROM). *Seismic Waves in Complex 3-D Structures*, **25**, 209–210, online at "http://sw3d.cz".

Bulant, P. & Klimeš, L. (2002): Numerical algorithm of the coupling ray theory in weakly anisotropic media. *Pure appl. Geophys.*, **159**, 1419–1435.

Bulant, P. & Klimeš, L. (2008): Numerical comparison of the isotropic-common-ray and anisotropic-common-ray approximations of the coupling ray theory. *Geophys. J. int.*, **175**, 357–374.

Klimeš, L. (2015): Determination of the reference symmetry axis of a generally anisotropic medium which is approximately transversely isotropic. *Seismic Waves in Complex 3-D Structures*, **25**, 177–185.

Klimeš, L. & Bulant, P. (2004): Errors due to the common ray approximations of the coupling ray theory. *Stud. geophys. geod.*, **48**, 117–142.

Klimeš, L. & Bulant, P. (2012): Single-frequency approximation of the coupling ray theory. *Seismic Waves in Complex 3-D Structures*, **22**, 143–167.

Klimeš, L. & Bulant, P. (2014): Prevailing-frequency approximation of the coupling ray theory for S waves along the SH and SV reference rays in a transversely isotropic medium. *Seismic Waves in Complex 3-D Structures*, **24**, 165–177.

Klimeš, L. & Bulant, P. (2015): Ray tracing and geodesic deviation of the SH and SV reference rays in a heterogeneous generally anisotropic medium which is approximately transversely isotropic. *Seismic Waves in Complex 3-D Structures*, **25**, 187–208.

Pšenčík, I. & Dellinger, J. (2001): Quasi-shear waves in inhomogeneous weakly anisotropic media by the quasi-isotropic approach: A model study. *Geophysics*, **66**, 308–319.

Pšenčík, I., Farra, V. & Tessmer, E. (2012): Comparison of the FORT approximation of the coupling ray theory with the Fourier pseudospectral method. *Stud. geophys. geod.*, **56**, 35–64.

Shearer, P.M. & Chapman, C.H. (1989): Ray tracing in azimuthally anisotropic media — I. Results for models of aligned cracks in the upper crust. *Geophys. J.*, **96**, 51–64.

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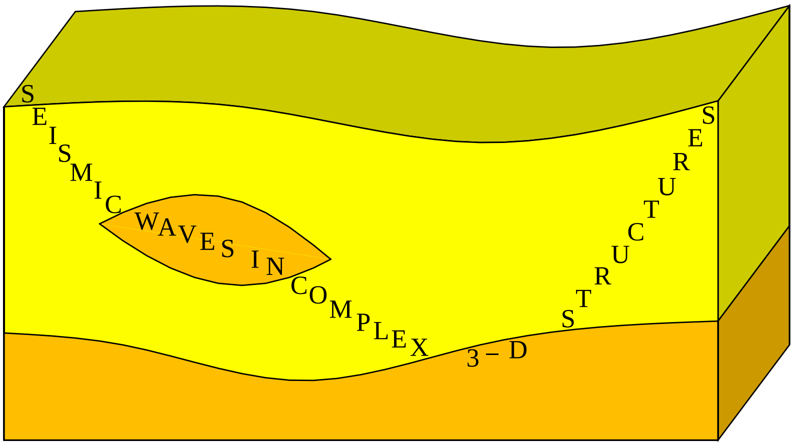

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