Calculation of the spatial gradient of the independent parameter along geodesics for a general Hamiltonian function

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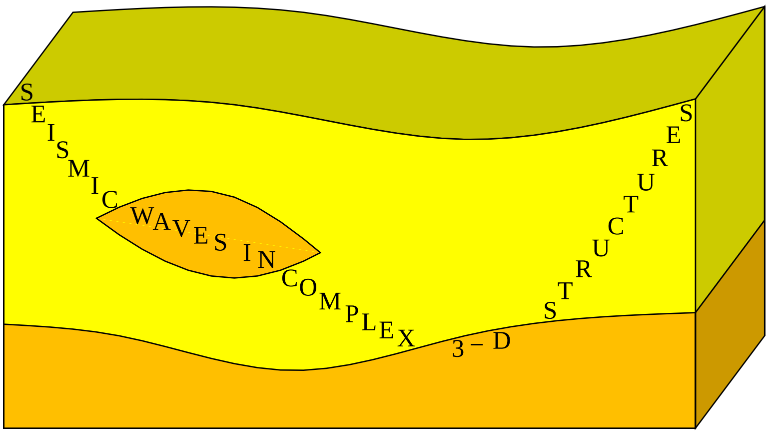

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Characteristic function

(point–to–point distance, two–point travel time)

The Characteristic function from point $\tilde{x}^n$ to point x^m:

$$V\left(x^m, \tilde{x}^n\right) \quad .$$

The characteristic function satisfies the Hamilton–Jacobi equations

$$H\Big(x^m, \frac{\partial V}{\partial x^n}\left(x^a, \tilde{x}^b\right)\Big) = C$$

and

$$H\Big(\tilde{x}^m, -\frac{\partial V}{\partial \tilde{x}^n}\left(x^a, \tilde{x}^b\right)\Big) = C \quad .$$

Equations of geodesics

Hamilton's equations
(equations of geodesics, equations of rays, ray tracing equations):

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad ,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}\gamma} = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad .$$

Hamilton's equations define function

$$\gamma(x^m, \tilde{x}^n)$$

from point $\tilde{x}^n$ to point x^m, with initial conditions $\gamma(\tilde{x}^m, \tilde{x}^n) = 0$.

Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a:

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \dfrac{\partial x^i}{\partial \tilde{x}^j} & \dfrac{\partial x^i}{\partial \tilde{y}_j} \\ \dfrac{\partial y_i}{\partial \tilde{x}^j} & \dfrac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad ,$$

where the derivatives of final point x^i and final slowness vector y_i with respect to initial point $\tilde{x}^j$ and initial slowness vector $\tilde{y}_j$ are taken at fixed parameter γ along geodesics (rays).

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function (Klimeš, 2013a)

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma}\frac{\partial \gamma}{\partial x^i}\frac{\partial \gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\Gamma}\frac{\partial \gamma}{\partial \tilde{x}^i}\frac{\partial \gamma}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma}\frac{\partial \gamma}{\partial \tilde{x}^j}\frac{\partial \gamma}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad ,$$

where integral

$$\Gamma = \int_0^\gamma \left(\frac{\partial \gamma}{\partial x^r}\frac{\partial^2 H}{\partial y_r \partial y_s}\frac{\partial \gamma}{\partial x^s}\right) \mathrm{d}\gamma$$

is calculated along the geodesic. Kronecker delta δ_i^k represents the components of the identity matrix.

How to calculate $\partial\gamma/\partial x^i$, $\partial\gamma/\partial\tilde{x}^i$ and Γ ?

Hamilton's (1837) identities

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} \, \frac{\partial H}{\partial y_j} = 0 \quad ,$$

$$\frac{\partial H}{\partial \tilde{y}_i} \, \frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} = 0 \quad .$$

Denote

$$X^{i\tilde{a}} = \frac{\partial x^i}{\partial \tilde{y}_a} \quad .$$

Matrix $X_{\tilde{a}k}$ inverse to matrix $X^{i\tilde{a}}$:

$$X^{i\tilde{a}} X_{\tilde{a}k} = \delta^i_k \quad .$$

Identity by Klimeš (2013a):

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^i} \frac{\partial \gamma}{\partial x^j} \right) = -X_{\tilde{i}j} \quad .$$

Multiplication by $\partial H / \partial \tilde{y}_i$ yields

$$\frac{\partial \gamma}{\partial x^j} = \Gamma \frac{\partial H}{\partial \tilde{y}_i} X_{\tilde{i}j} \quad .$$

Multiplication by $\partial H / \partial y_j$ yields

$$\frac{\partial \gamma}{\partial \tilde{x}^i} = -\Gamma X_{\tilde{i}j} \frac{\partial H}{\partial y_j} \quad .$$

Multiplication by both $\partial H / \partial \tilde{y}_i$ and $\partial H / \partial y_j$ yields

$$\Gamma = \left(\frac{\partial H}{\partial \tilde{y}_i} X_{\tilde{i}j} \frac{\partial H}{\partial y_j} \right)^{-1} \quad .$$

References

Hamilton, W.R. (1837): Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144.

Klimeš, L. (2013a): Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function. In: *Seismic Waves in Complex 3-D Structures, Report 22*, pp. 121–134, Dep. Geophys., Charles Univ., Prague, online at "http://sw3d.cz".

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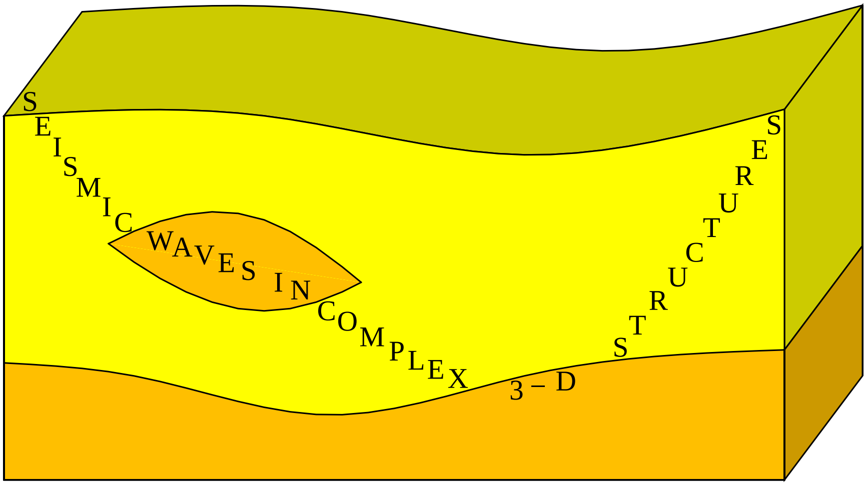

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