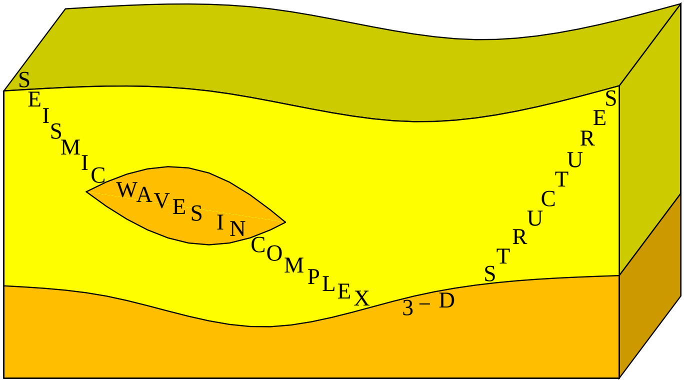
Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function

Luděk Klimeš

Department of Geophysics
Faculty of Mathematics and Physics
Charles University in Prague



http://sw3d.cz

Characteristic function

(point–to–point distance, two–point travel time)

The Characteristic function from point $\tilde{x}^n$ to point x^m:

$$V(x^m, \tilde{x}^n) \quad .$$

The characteristic function satisfies the Hamilton–Jacobi equations

$$H\Big(x^m, \frac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\Big) = C$$

and

$$H\Big(\tilde{x}^m, -\frac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\Big) = C \quad .$$

Equations of geodesics

Hamilton's equations
(equations of geodesics, equations of rays, ray tracing equations):

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad ,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}\gamma} = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad .$$

Hamilton's equations define function

$$\gamma(x^m, \tilde{x}^n)$$

from point $\tilde{x}^n$ to point x^m, with initial conditions $\gamma(\tilde{x}^m, \tilde{x}^n) = 0$.

Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a:

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \dfrac{\partial x^i}{\partial \tilde{x}^j} & \dfrac{\partial x^i}{\partial \tilde{y}_j} \\ \dfrac{\partial y_i}{\partial \tilde{x}^j} & \dfrac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} ,$$

where the derivatives of final point x^i and final slowness vector y_i with respect to initial point $\tilde{x}^j$ and initial slowness vector $\tilde{y}_j$ are taken at fixed parameter γ along geodesics (rays).

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a homogeneous Hamiltonian function of the second degree (Klimeš, 2009)

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j} \left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^j} \frac{\partial V}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad .$$

Kronecker delta δ_i^k represents the components of the identity matrix.

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function (Klimeš, 2013)

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial x^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j} \left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^j} \frac{\partial \gamma}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad ,$$

where integral

$$\Gamma = \int_0^\gamma \left(\frac{\partial \gamma}{\partial x^r} \frac{\partial^2 H}{\partial y_r \partial y_s} \frac{\partial \gamma}{\partial x^s} \right) \mathrm{d}\gamma$$

is calculated along the geodesic. Kronecker delta δ_i^k represents the components of the identity matrix.

The relations are not applicable if integral Γ is equal to zero, which may happen, e.g., if Hamiltonian function $H(x^i, y_j)$ is a homogeneous function of the first degree with respect to y_n.

The relations are applicable to the spatial ray methods but not to the space-time ray methods.

References (online at "http://sw3d.cz")

Klimeš, L. (2009): Relation between the propagator matrix of geodesic deviation and second–order derivatives of characteristic function. In: *Seismic Waves in Complex 3–D Structures, Report 19*, pp. 103–114, Dep. Geophys., Charles Univ., Prague.

Klimeš, L. (2013): Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function. In: *Seismic Waves in Complex 3-D Structures, Report 22*, pp. 121–134, Dep. Geophys., Charles Univ., Prague.

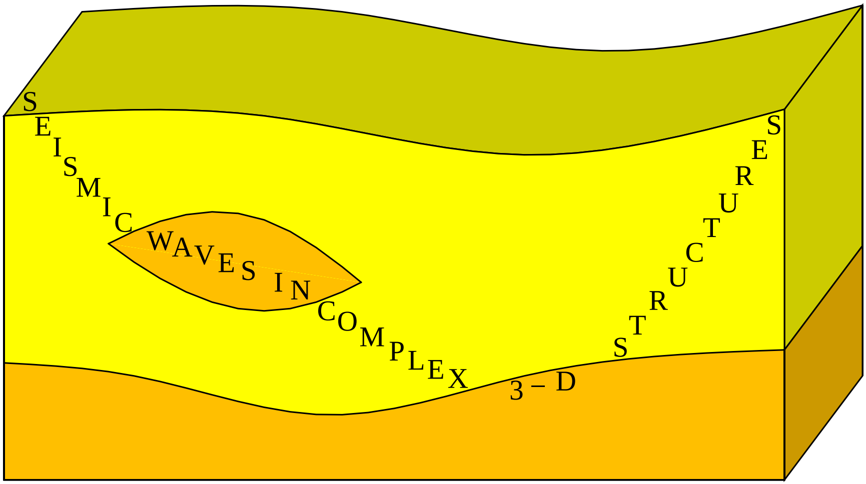
Acknowledgements

The research has been supported:

by the Grant Agency of the Czech Republic under contract P210/10/0736,

by the Ministry of Education of the Czech Republic within research project MSM0021620860,

and by the consortium “Seismic Waves in Complex 3-D Structures”



http://sw3d.cz