Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function

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**Characteristic function**  
(point–to–point distance, two–point travel time)

The Characteristic function from point $\tilde{x}^n$ to point $x^m$:

$$V(x^m, \tilde{x}^n).$$

The characteristic function satisfies the Hamilton–Jacobi equations

$$H(x^m, \frac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)) = C$$

and

$$H(\tilde{x}^m, -\frac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)) = C.$$
Equations of geodesics

Hamilton’s equations
(equations of geodesics, equations of rays, ray tracing equations):

\[
\frac{dx^i}{d\gamma} = \frac{\partial H}{\partial y_i}(x^m, y_n),
\]

\[
\frac{dy_i}{d\gamma} = -\frac{\partial H}{\partial x^i}(x^m, y_n).
\]

Hamilton’s equations define function

\[\gamma(x^m, \tilde{x}^n)\]

from point \(\tilde{x}^n\) to point \(x^m\), with initial conditions \(\gamma(\tilde{x}^m, \tilde{x}^n) = 0\).
Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point $x^a$:

$$
\Pi(x^a, \tilde{x}^b) = \begin{pmatrix}
\frac{\partial x^i}{\partial \tilde{x}^j} & \frac{\partial x^i}{\partial \tilde{y}_j} \\
\frac{\partial y_i}{\partial \tilde{x}^j} & \frac{\partial y_i}{\partial \tilde{y}_j}
\end{pmatrix},
$$

where the derivatives of final point $x^i$ and final slowness vector $y_i$ with respect to initial point $\tilde{x}^j$ and initial slowness vector $\tilde{y}_j$ are taken at fixed parameter $\gamma$ along geodesics (rays).
Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a homogeneous Hamiltonian function of the second degree (Klimeš, 2009)

\[
\left( \frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial x^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} \frac{\partial}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} ,
\]

\[
\left( \frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^i} \frac{\partial V}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta^k_i ,
\]

\[
\frac{\partial x^i}{\partial \tilde{y}_j} \left( \frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{V} \frac{\partial V}{\partial \tilde{x}^j} \frac{\partial V}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k} .
\]

Kronecker delta \( \delta^k_i \) represents the components of the identity matrix.
Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a general Hamiltonian function (Klimeš, 2013)

\[
\left( \frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial x^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k},
\]

\[
\left( \frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^i} \frac{\partial \gamma}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta^k_i,
\]

\[
\frac{\partial x^i}{\partial \tilde{y}_j} \left( \frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\Gamma} \frac{\partial \gamma}{\partial \tilde{x}^j} \frac{\partial \gamma}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k},
\]

where integral

\[
\Gamma = \int_0^\gamma \left( \frac{\partial \gamma}{\partial x^r} \frac{\partial^2 H}{\partial y_r \partial y_s} \frac{\partial \gamma}{\partial x^s} \right) d\gamma
\]

is calculated along the geodesic. Kronecker delta $\delta^k_i$ represents the components of the identity matrix.
The relations are not applicable if integral $\Gamma$ is equal to zero, which may happen, e.g., if Hamiltonian function $H(x^i, y_j)$ is a homogeneous function of the first degree with respect to $y_n$.

The relations are applicable to the spatial ray methods but not to the space-time ray methods.

Acknowledgements

The research has been supported:
by the Grant Agency of the Czech Republic under contract P210/10/0736,
by the Ministry of Education of the Czech Republic within research project MSM0021620860,
and by the consortium “Seismic Waves in Complex 3-D Structures”

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