Gaussian beams
in inhomogeneous layered anisotropic media
(DRT in Cartesian coordinates)

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Outline:

Introduction
Dynamic ray tracing
Transformation relation
Paraxial approximation
Gaussian beams
Conclusions
Introduction

Gaussian beams - solutions of elastodyn. equation concentrated to ray

exact solutions $\Rightarrow$ Gaussian beams

approximate solutions $\Rightarrow$ paraxial Gaussian beams

Gaussian beams in isotropic media - broad applications

Gaussian beams in anisotropic media - Hanyga, 1986; Alkhalifah, 1995

ray tracing (RT) $\Rightarrow$ central ray $\Omega$

dynamic ray tracing (DRT) $\Rightarrow$ amplitudes,

second derivatives of traveltime along $\Omega$
Dynamic ray tracing

DRT in ray-centred coordinates (last year)

DRT in Cartesian coordinates (now)

  DRT in global Cartesian coordinates $x_i$
  DRT in local Cartesian (WOC) coordinates $y_i$

simplified DRT in global Cartesian coordinates $x_i$

Solution of DRT

  using ray propagator matrix along $\Omega$

    - plane-wave and point-source initial conditions

  specified initial conditions - real-valued $\times$ complex-valued
DRT in Cartesian coordinates

6 linear ordinary differential equations of first order
simple structure of ordinary differential equations
ray propagator matrix $6 \times 6 \Rightarrow$ solution of DRT $6 \times$
evaluation of GB at any point in a vicinity of a point on $\Omega$
simplified DRT - ray propagator matrix $4 \times 4$

$\Rightarrow$ solution of DRT $4 \times +$ transformation

DRT in ray-centred coordinates

4 linear ordinary differential equations of first order
more complicated structure of ordinary differential equations
ray propagator matrix $4 \times 4 \Rightarrow$ solution of DRT $4 \times$
evaluation of GB only in plane tangent to wavefront

at its intersection with $\Omega$
Transformation of 2nd traveltime derivatives

\[ M^{(x)} = f M^{(y)} f^T + p \eta^T + \eta p^T - pp^T(\mathcal{U}^T \eta) \]

- \( M^{(x)} \) - 6 \times 6 matrix with \( \partial^2 T / \partial x_i \partial x_j \)
- \( M^{(y)} \) - 4 \times 4 matrix with \( \partial^2 T / \partial y_N \partial y_M \)
- \( p \) - slowness vector
- \( \mathcal{U} \) - ray-velocity vector
- \( \eta \) - eta vector
- \( f = (f_1, f_2) \) - \( f_K \) vectors perpendicular to ray \( \Omega \)
- \( p, \mathcal{U}, \eta \) obtained from tracing ray \( \Omega \)
- \( f_K \) - covariant basis vectors in ray-centred coordinates
- (require solution of one vectorial differential equation along \( \Omega \))
Paraxial approximation of displacement vector (WOC)

Initial conditions: $2 \times 2$ \textbf{real-valued} matrix $M^y(\tau_0)$

$$u(y, \tau) = U^\Omega(\tau)[\text{det } Q^y(\tau_0)/\text{det } Q^y(\tau)]^{1/2} \exp[-i\omega(t - T(\tau) - \frac{1}{2}y^T M^y(\tau) y)]$$

$u(y, \tau)$ - displacement vector

$U^\Omega(\tau)$ - spreading-free vectorial amplitude

$\text{det } Q^y(\tau_0)/\text{det } Q^y(\tau)$, $M^y(\tau)$ - \textbf{real valued}, obtained from ray propagator matrix and $M^y(\tau_0)$ specified at $\tau = \tau_0$

$T$ - traveltime
Gaussian beams (WOC)

Initial conditions: 2 × 2 complex-valued matrix $M^y(\tau_0)$

Existence conditions:

$$M^y(\tau_0)$$ is symmetric and finite, $\text{Im}M^y(\tau_0)$ is positive definite

$$u(y, \tau) = U^\Omega(\tau)[\det Q^y(\tau_0)/\det Q^y(\tau)]^{1/2} \exp[-\frac{1}{2}\omega y^T \text{Im}M^y(\tau)y]$$

$$\times \exp[-i\omega(t - T(\tau) - \frac{1}{2}q^T \text{Re}M^y(\tau)q)]$$

$U^\Omega(\tau)$ - spreading-free vectorial amplitude

$\det Q^y(\tau_0)/\det Q^y(\tau)$, $M^y(\tau)$ - complex valued

obtained from ray propagator matrix and $M^y(\tau_0)$ specified at $\tau = \tau_0$

$T$ - traveltime
Conclusions

- various options for constructing GB in anisotropic media
- efficiency to be studied
- use of ray propagator matrix \( M^{(x)} \) at termination point of GB
- advantageous to use \( M^{(x)} \) at termination point of GB
- possible simplifications in specific cases