Historical introduction to perturbation and paraxial ray methods for anisotropic media

Transformation of spatial and perturbation derivatives of travel time at a general interface between two general media

Transformation of paraxial matrices at a general interface between two general media

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Paraxial ray methods for anisotropic media were established by William Rowan Hamilton in 1832 (Hamilton, 1837)

Slowness surface, ray-velocity surface.

Equations of rays (Hamilton's equations, ray-tracing equations, equations of geodesics).

Snell's law for anisotropic media.

Paraxial approximation of travel time (action).

Two-point paraxial approximation of two-point travel time (characteristic function).

Transformation of the second-order spatial derivatives of travel time at curved interfaces between generally anisotropic heterogeneous media.

Transformation of the second-order spatial derivatives of two-point travel time at curved interfaces between generally anisotropic heterogeneous media.

Properties of the second-order two-point spatial derivatives of two-point travel time.

Geodesic deviation and second-order derivatives of travel time in general coordinates

Hamiltonian equations of geodesic deviation (dynamic ray tracing equations):
Červený (1972).

Second-order spatial derivatives of travel time:
Gajewski & Pšenčík (1987).

Transformation of the paraxial matrices of orthonomic systems of rays at curved interfaces between generally anisotropic heterogeneous media:
Gajewski & Pšenčík (1990).

Transformation of the propagator matrix of geodesic deviation at curved interfaces between generally anisotropic heterogeneous media:
Farra & Le Bégat (1995), Klimeš (2010b).

Relation between the propagator matrix of geodesic deviation and the second-order two-point spatial derivatives of two-point travel time:
Klimeš (2009).

Perturbation and paraxial ray methods in general coordinates

Equations for calculating both spatial and perturbation derivatives of travel time of arbitrary orders along unperturbed rays:
Klimeš (2002).

Transformation of both spatial and perturbation derivatives of travel time of arbitrary orders at curved interfaces between generally anisotropic heterogeneous media:
Klimeš (2010a).

Perturbation Hamiltonian function for the perturbation expansion of complex-valued travel time along real-valued reference rays in attenuating media:
Klimeš & Klimeš (2010).

Ray-centred coordinates in anisotropic media

Special cases of ray-centred coordinates in anisotropic media:
Hanyga (1982), Kendall, Guest & Thomson (1992), Klimeš (2006).

Transformation relations for the Hamiltonian equations of geodesic deviation and for the propagator matrix of geodesic deviation between general coordinates and ray-centred coordinates:
Klimeš (1994).

Transformation relations for the second-order spatial derivatives of travel time between general coordinates and ray-centred coordinates:
Červený & Klimeš (2010).

Transformation of spatial and perturbation derivatives of travel time at a general interface between two general media

Smooth interface:

$$F(x^i, f^\alpha) = 0$$

x^i are spatial coordinates,
f^α are perturbation parameters,
$\widetilde{\tau} = \widetilde{\tau}(x^i, f^\alpha)$ is the incident travel time,
$\tau = \tau(x^i, f^\alpha)$ is the reflected or refracted travel time.

Snell's law:

$$\boxed{\tau_{,\underline{i}} = \widetilde{\tau}_{,\underline{i}} + F_{,\underline{i}}\,\lambda}$$

$F_{,\underline{i}}$ represents both $\dfrac{\partial F}{\partial x^i}$ and $\dfrac{\partial F}{\partial f^\alpha}$, analogously for $\widetilde{\tau}_{,\underline{i}}$ and $\tau_{,\underline{i}}$,
λ is the Lagrange multiplier calculated from the non-linear algebraic equation.

Transformation of spatial and perturbation derivatives of travel time at a general interface between two general media

$$\tau_{,\underline{i}\underline{j}} = \widetilde{\tau}_{,\underline{i}\underline{j}} + F_{,\underline{i}\underline{j}}\,\lambda + F_{,\underline{i}}\,\lambda_{\underline{j}} + F_{,\underline{j}}\,\lambda_{\underline{i}}$$

$$\tau_{,\underline{i}\underline{j}\underline{k}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}} + F_{,\underline{i}\underline{j}\underline{k}}\,\lambda + F_{,\underline{i}\underline{j}}\,\lambda_{\underline{k}} + F_{,\underline{i}\underline{k}}\,\lambda_{\underline{j}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i}} + F_{,\underline{i}}\lambda_{\underline{j}\underline{k}} + F_{,\underline{j}}\lambda_{\underline{i}\underline{k}} + F_{,\underline{k}}\lambda_{\underline{i}\underline{j}}$$

$$\begin{aligned}\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}\underline{l}} &+ F_{,\underline{i}\underline{j}\underline{k}\underline{l}}\,\lambda \\ &+ F_{,\underline{i}\underline{j}\underline{k}}\,\lambda_{\underline{l}} + F_{,\underline{i}\underline{j}\underline{l}}\,\lambda_{\underline{k}} + F_{,\underline{i}\underline{k}\underline{l}}\,\lambda_{\underline{j}} + F_{,\underline{j}\underline{k}\underline{l}}\,\lambda_{\underline{i}} \\ &+ F_{,\underline{i}\underline{j}}\,\lambda_{\underline{k}\underline{l}} + F_{,\underline{i}\underline{k}}\,\lambda_{\underline{j}\underline{l}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i}\underline{l}} + F_{,\underline{i}\underline{l}}\,\lambda_{\underline{j}\underline{k}} + F_{,\underline{j}\underline{l}}\,\lambda_{\underline{i}\underline{k}} + F_{,\underline{k}\underline{l}}\,\lambda_{\underline{i}\underline{j}} \\ &+ F_{,\underline{i}}\,\lambda_{\underline{j}\underline{k}\underline{l}} + F_{,\underline{j}}\,\lambda_{\underline{i}\underline{k}\underline{l}} + F_{,\underline{k}}\,\lambda_{\underline{i}\underline{j}\underline{l}} + F_{,\underline{l}}\,\lambda_{\underline{i}\underline{j}\underline{k}}\end{aligned}$$

$\lambda_{\underline{i}}$, $\lambda_{\underline{i}\underline{j}}$, $\lambda_{\underline{i}\underline{j}\underline{k}}$ are the Lagrange multipliers calculated using the explicit expressions.

Transformation of paraxial matrices at a general interface between two general media

Detailed derivation of the equations by Farra & Le Bégat (1995). Discussion of the difference from the derivation of Gajewski & Pšenčík (1990).

Coordinates of points of rays:

$$x^i = x^i(\gamma^{\underline{a}}, \gamma) \quad . \tag{10}$$

Corresponding components of slowness vectors:

$$p_i = p_i(\gamma^{\underline{a}}, \gamma) \quad . \tag{11}$$

$\gamma^{\underline{a}}$ ... initial parameters parametrizing the initial conditions for rays.

γ ... parameter along rays determined by the form of the Hamiltonian function.

Paraxial matrices:

$$Q^{i}{}_{\underline{a}} = \frac{\partial x^{i}}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad , \tag{12}$$

$$P_{i\underline{a}} = \frac{\partial p_{i}}{\partial \gamma^{\underline{a}}}(\gamma^{\underline{c}}, \gamma) \quad . \tag{13}$$

Propagator matrix of geodesic deviation from point x_0^n to point x^m:

$$\mathbf{\Pi}(x^m, x_0^n) = \begin{pmatrix} \frac{\partial x^i}{\partial x_0^j} & \frac{\partial x^i}{\partial p_j^0} \\ \frac{\partial p_i}{\partial x_0^j} & \frac{\partial p_i}{\partial p_j^0} \end{pmatrix} \quad . \tag{16}$$

These matrices are the solution of the Hamiltonian equations of geodesic deviation (dynamic ray tracing equations).

These matrices must be transformed at interfaces.

Smooth interface:

$$F(x^i) = 0 \quad . \tag{21}$$

Hamiltonian function corresponding to the incident travel time:

$$\widetilde{H}\big(x^i, \widetilde{\tau}_{,j}(x^m)\big) = \widetilde{C} \quad . \tag{22}$$

Hamiltonian function corresponding to the reflected or refracted travel time:

$$H\big(x^i, \tau_{,j}(x^m)\big) = C \quad . \tag{23}$$

Phase-space derivatives of the Hamiltonian function:

$$\widetilde{H}_{,i} = \frac{\partial \widetilde{H}}{\partial x^i} \quad , \quad \widetilde{H}^{,a} = \frac{\partial \widetilde{H}}{\partial p^a} \quad , \quad H_{,i} = \frac{\partial H}{\partial x^i} \quad , \quad H^{,a} = \frac{\partial H}{\partial p^a} \quad . \tag{4}$$

Transformation of paraxial matrices at the interface:

$$\begin{pmatrix} Q^i{}_{\underline{a}} \\ P_{i\underline{a}} \end{pmatrix} = \begin{pmatrix} C^i{}_j & 0^{ij} \\ D_{ij} & E_i{}^j \end{pmatrix} \begin{pmatrix} \widetilde{Q}^j{}_{\underline{a}} \\ \widetilde{P}_{j\underline{a}} \end{pmatrix} \quad . \tag{82}$$

Transformation of the propagator matrix of geodesic deviation:

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{pmatrix} \widetilde{\mathbf{\Pi}} \quad . \tag{83}$$

Matrix $\left(\begin{smallmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{smallmatrix}\right)$ is often called the interface propagator matrix.

$$C^i{}_j = \delta^i_j + \left(H^{,i} - \widetilde{H}^{,i}\right) \widetilde{N}_j \quad . \tag{57}$$

$$\begin{aligned} D_{ij} = {} & \lambda \left(\delta^k_i - N_i H^{,k}\right) F_{,kl} \left(\delta^l_j - \widetilde{H}^{,l} \widetilde{N}_j\right) \\ & + N_i \left(\widetilde{H}_{,k} - H_{,k}\right) \left(\delta^k_j - \widetilde{H}^{,k} \widetilde{N}_j\right) + \left(\delta^k_i - N_i H^{,k}\right) \left(\widetilde{H}_{,k} - H_{,k}\right) \widetilde{N}_j \\ & + N_i \left(\widetilde{H}^{,r} \widetilde{H}_{,r} - H^{,r} H_{,r}\right) \widetilde{N}_j \quad . \end{aligned} \tag{80}$$

$$E_i{}^j = \delta^j_i + N_i \left(\widetilde{H}^{,j} - H^{,j}\right) \tag{79}$$

$$\widetilde{N}_i = \left(\widetilde{H}^{,q} F_{,q}\right)^{-1} F_{,i} \quad , \tag{52}$$

$$N_i = \left(H^{,q} F_{,q}\right)^{-1} F_{,i} \quad . \tag{53}$$

0^{ij} are the components of the zero matrix.

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