SH-wave reflection coefficient in an isotropic, weakly attenuating medium

Miłosz Wcisło (IRSM CAS, MFF CUNI, Prague, Czech Republic)
Ivan Pšenčík (IG CAS, Prague, Czech Republic)
Presentation Outline

• Weak Attenuation Concept (WAC)
• Application of the WAC
• Slowness vectors
• Reflection coefficient in WAC
• Numerical tests
• Conclusions
Weak Attenuation Concept

Anelastic medium can be described by complex-valued stiffness tensor (correspondence principle):

\[ a_{ijkl}(\omega) = a^R_{ijkl} + i a^I_{ijkl} \]

**complex ray tracing is difficult!**

- In case \( a^I_{ijkl} \ll a^R_{ijkl} \), it is possible to treat the imaginary part of the stiffness tensor as a perturbation of its real part.
Weak Attenuation Concept

Gajewski & Pšenčík (1992) applied the WAC inside layers. They showed that the effect of attenuation can be expressed as:

\[ t^* = \int_{\tau_0}^{\tau} a_{ijkl} p_j p_l g_i g_k d\tau^R \]

where:
- \( t^* \) - global absorption factor
- \( p \) - slowness vector
- \( g \) - unit polarization vector
- \( \tau^R \) - traveltime along the ray in the reference elastic medium

It was shown that the approximation works for \( Q > 50 \).
Application of the WAC

Elastic case:

Reflection coefficient:

\[ R = \frac{\rho_1 \beta_1^2 p_i N_i - \rho_2 \beta_2^2 p_k^{(t)} N_k}{\rho_1 \beta_1^2 p_i N_i + \rho_2 \beta_2^2 p_k^{(t)} N_k} \]

- \( p \) - density
- \( \beta \) - SH - wave velocity

All quantities are real valued!
Application of the WAC

Anelastic case

Reflection coefficient:

\[ R = \frac{\rho_1 \beta_1^2 p_i N_i - \rho_2 \beta_2^2 p_k^{(t)} N_k}{\rho_1 \beta_1^2 p_i N_i + \rho_2 \beta_2^2 p_k^{(t)} N_k} \]

In anelastic case we consider:

\[ \beta_i(\omega) = \beta_i^R(\omega) \left[ 1 - \frac{1}{2} i Q_i^{-1}(\omega) \right] \]
\[ p_i = P_i + i A_i \]
\[ p_i^{(t)} = P_i^{(t)} + i A_i^{(t)} \]
Applicability of the WAC

We can apply WAC if:

\( Q^{-1} \) – attenuation coefficient

\( D(\gamma) \) – inhomogeneity factor (angle)

\[ D \text{ - inhomogeneity factor} \]
\[ m \text{ - unit vector perpendicular to the propagation vector } P \]
Slowness vectors

Incident wave:

\[ p_i = P_i + iA_i = P_i + i\left(\frac{1}{2} Q_{1^{-1}} P_i + Dm_i\right) \]

- **P** – propagation vector
- **A** – attenuation vector
- **D** – inhomogeneity factor
- **m** – unit vector perpendicular to **P**
Slowness vectors

Slowness vectors at the interface need to satisfy:

1. relations resulting from the approximate complex-valued eikonal equation
   \[ p_i p_i = \beta_1^{-2}, \quad p_i A_i = \frac{1}{2} \beta_1^{-2} Q_1^{-1}, \quad p_i^{(t)} p_i^{(t)} = \beta_2^{-2}, \quad p_i^{(t)} A_i^{(t)} = \frac{1}{2} \beta_2^{-2} Q_2^{-1} \]

2. the Snell law - the tangential components of complex-valued slowness vectors of the incident and generated waves are equal
   \[ p_i^{(t)} - N_i \left( p_k^{(t)} N_k \right) = p_i - N_i (p_k N_k) \]

3. radiation condition implicitly in the reference wave
Slowness vector of transmitted wave

Subcritical incidence:

Propagation vector:

\[ P_i^{(t)} = P_i - N_i \beta_1^{-1} X_1 + N_i \beta_2^{-1} X_2 \]
\[ X_1 = (1 - \beta_1^2 p^2)^{1/2} \]
\[ X_2 = (1 - \beta_2^2 p^2)^{1/2} \]
\[ p \] - ray parameter

Attenuation vector:

\[ A_i^{(t)} = A_i - N_i \beta_1^{-1} \xi + N_i \beta_2^{-1} \xi^{(t)} \]
\[ \xi = \frac{1}{2} Q_1^{-1} X_1 + \beta_1 Dm_i N_i \]
\[ \xi^{(t)} = \frac{1}{2} ZX_2^{-1} \]
\[ Z = Q_2^{-1} - r^2 Q_1^{-1} + 2X_1 r^2 \xi \]
\[ r = \beta_2 / \beta_1 \]

\[ Z \] controls orientation of the attenuation vector
**Slowness vector of transmitted wave**

**Overcritical incidence:**

**Propagation vector:**

\[
P_i^{(t)} = P_i - N_i \beta_1^{-1} X_1 + N_i \beta_2^{-1} \bar{\xi}^{(t)}
\]

\[
X_1 = (1 - \beta_1^2 p^2)^{1/2}
\]

\[
\bar{\xi}^{(t)} = \frac{1}{2} Z \bar{X}_2^{-1}
\]

\[
Z = Q_2^{-1} - r^2 Q_1^{-1} + 2X_1 r^2 \xi
\]

\[
p \quad \text{ray parameter}
\]

**Attenuation vector:**

\[
A_i^{(t)} = A_i - N_i \beta_1^{-1} \xi + N_i \beta_2^{-1} \bar{X}_2
\]

\[
X_2 = i \bar{X}_2 = i (\beta_2^2 p^2 - 1)^{1/2}
\]

\[
\xi = \frac{1}{2} Q_1^{-1} X_1 + \beta_1 D m_i N_i
\]

\[
r = \beta_2 / \beta_1
\]

\[Z \] controls orientation of the propagation vector
Slowness vector of transmitted wave

Model:

\[ \beta_1 = 3.698 \text{ km/s} \]
\[ \beta_2 = 4.618 \text{ km/s} \]
\[ \rho_1 = 2.98 \text{ kg/m}^3 \]
\[ \rho_2 = 3.30 \text{ kg/m}^3 \]
\[ Q_1 = 75 \]
\[ Q_2 = 50 \]

Homogeneous incident wave
Slowness vector of transmitted wave

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- \( \beta_2 = 4.618 \text{ km/s} \)
- \( \rho_1 = 2.98 \text{ kg/m}^3 \)
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Homogeneous incident wave
WAC reflection coefficient

\[ R = \frac{\rho_1 \beta_1^2 (1 - iQ_1^{-1})(P_i + iA_i)N_i}{\rho_1 \beta_1^2 (1 - iQ_1^{-1})(P_i + iA_i)N_i} - \frac{\rho_2 \beta_2^2 (1 - iQ_2^{-1})(P_i^{(t)} + iA_i^{(t)})N_i}{\rho_1 \beta_1^2 (1 - iQ_1^{-1})(P_i + iA_i)N_i} + \frac{\rho_2 \beta_2^2 (1 - iQ_2^{-1})(P_i^{(t)} + iA_i^{(t)})N_i}{\rho_1 \beta_1^2 (1 - iQ_1^{-1})(P_i + iA_i)N_i} \]
Numerical tests + benchmarks

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Homogeneous incident wave

Numerical tests + benchmarks (testing limits)

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\[ Q_1 = 40, 30, 20 \]
\[ Q_2 = 50 \]

Homogeneous incident wave

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- $\beta_1 = 3.698 \text{ km/s}$
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- $Q_2 = 40, 30, 20$

Homogeneous incident wave

red - WAC

green - Brokesova & Cerveny (1998)

Numerical tests + benchmarks (testing limits)

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Inhomogeneous incident wave

Numerical tests + benchmarks (testing limits)

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\[ Q_1 = 50 \]
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Inhomogeneous incident wave

red - WAC

green - Brokesova & Cerveny (1998)

Conclusions

• weak-attenuation concept leads to simple and transparent formulae

• subcritical incidence - attenuation vector may point up

• overcritical incidence - propagation vector may point up

• acceptable results even for moderate attenuation and moderate wave inhomogeneity

• WAC fails in the vicinity of the critical angle

• effect of attenuation on reflection seems to be weaker than on propagation in layers
Plans

• tests of accuracy by comparing with independent methods
• transmission coefficient
• ray synthetic seismograms in layered attenuative media
• application to anisotropic media
Bonus slides
Transmission coefficient in WAC

\[ T = \frac{2 \rho_1 \beta_1^2 p_i N_i}{\rho_1 \beta_1^2 p_i N_i + \rho_2 \beta_2^2 p_k^{(t)} N_k} \]
Numerical tests + benchmarks

Model:
\[
\beta_1 = 3.698 \text{ km/s} \\
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Homogeneous incident wave

Numerical tests + benchmarks

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Homogeneous incident wave