Hamiltonian formulation of the Finsler geometry and the relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function

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Finsler geometry

At each point x^m, Finslerian metric function $F(x^m, y^n)$ is a positively homogeneous function of the first degree in the tangent space $\{y^i\}$.
It defines the infinitesimal distance (travel-time) $\mathrm{d}V = F(x^m, \mathrm{d}x^n)$.

Characteristic function (point-to-point distance, two-point travel time) $V(x^m, \tilde{x}^n)$ from point $\tilde{x}^n$ to point x^m is the stationary value of path integral $\mathcal{V}(x^m, \tilde{x}^n) = \int_0^t F\left[x^i(t), \frac{\mathrm{d}x^j}{\mathrm{d}t}(t)\right] \mathrm{d}t$ from point $\tilde{x}^n$ to point x^m.

In turn, the Finslerian metric function is the Gâteaux variation
$F(x^m, y^n) = \lim_{t \to 0+} \left(\frac{1}{t} V(x^m + t\, y^m, x^n)\right)$
of the Characteristic function in the direction of y^i.

Hamiltonian formulation of the Finsler geometry

The Hamiltonian formulation of the Finsler geometry is based on the Hamiltonian function $H(x^m, y_n)$ which defines the figuratrix (phase-slowness surface) $H(x^m, y_n) = C$ in the cotangent space $\{y_i\}$, and on the Hamilton-Jacobi equations

$$H\left(x^m, \frac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\right) = C \quad , \qquad H\left(\tilde{x}^m, -\frac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\right) = C$$

for the characteristic function (point-to-point distance, two-point travel time) $V(x^m, \tilde{x}^n)$ from point $\tilde{x}^n$ to point x^m.

The Finslerian formulation of the Finsler geometry is based on the Finslerian metric function $F(x^m, y^n)$ which defines the indicatrix (ray-velocity surface) $F(x^m, y^n) = 1$ in the tangent space $\{y^i\}$.

If both the Hamiltonian and Finslerian formulations are applicable, they are equivalent and may be applied simultaneously.
We can obtain $F(x^m, y^n)$ from $H(x^m, y_n)$ or $H(x^m, y_n)$ from $F(x^m, y^n)$.

However, there are situations in which the Hamiltonian formulation has some advantages.

(a) The Hamiltonian equations of geodesics are simpler than the Finslerian equations.

The Hamiltonian equations of geodesics contain the first-order phase-space derivatives of the Hamiltonian function, whereas the Finslerian equations of geodesics contain the third-order phase-space derivatives of the Hamiltonian function in Christoffel symbols $\Gamma^i_{\;jk}$.

Hamilton's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} x^i = \frac{\partial H}{\partial y_i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y_i = -\frac{\partial H}{\partial x^i}$$

Finslerian equations of geodesics:

$$\frac{\mathrm{d}}{\mathrm{d}t} x^i = y^i$$

$$\frac{\mathrm{d}}{\mathrm{d}t} y^i = -\Gamma^i_{\;jk}\, y^j\, y^k$$

(b) The Hamiltonian equations of geodesic deviation are considerably simpler than the Finslerian equations.

The Hamiltonian equations of geodesic deviation for derivatives $X^i = \frac{\partial x^i}{\partial \gamma}$ and $Y_i = \frac{\partial y_i}{\partial \gamma}$ with respect to the initial conditions parametrized by γ contain the second-order phase-space derivatives of the Hamiltonian function, whereas the Finslerian equations contain the fourth-order phase-space derivatives of the Hamiltonian function in curvature tensor R_{ijkl}.

Hamiltonian eqs. of geodesic dev.:

$$\frac{\mathrm{d}}{\mathrm{d}t} X^i = \frac{\partial^2 H}{\partial y_i \partial x^j} X^j + \frac{\partial^2 H}{\partial y_i \partial y_j} Y_j$$

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_i = -\frac{\partial^2 H}{\partial x^i \partial x^j} X^j - \frac{\partial^2 H}{\partial x^i \partial y_j} Y_j$$

Finslerian eqs. of geodesic deviation:

$$\frac{\mathrm{d}}{\mathrm{d}t} X^i = -\Gamma^i_{sj}\, y^s X^j + \tilde{Y}^i$$

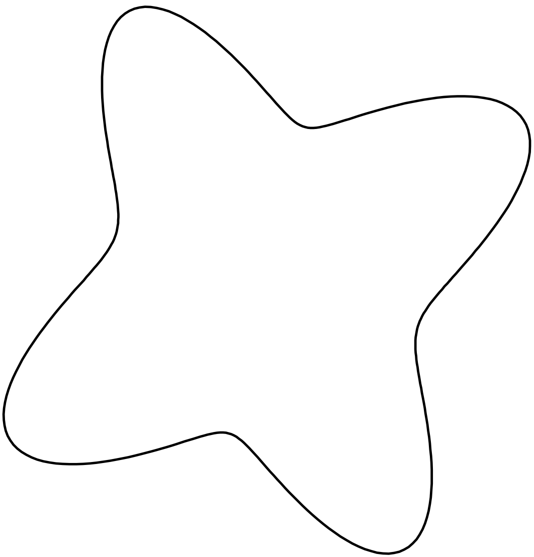
$$\frac{\mathrm{d}}{\mathrm{d}t} \tilde{Y}^i = g^{ik} R_{ksrj}\, y^s y^r X^j - \Gamma^i_{sj}\, y^s \tilde{Y}^j$$

$$\tilde{Y}^i = g^{ir}\left(Y_r - \Gamma^s_{rj}\, y_s X^j\right)$$

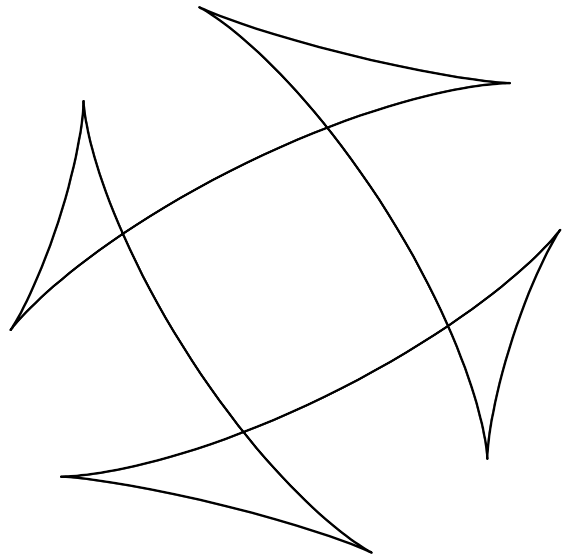
(c) In the high-frequency approximation of wave propagation (geometrical optics), the Finsler geometry is generated by the Hamilton-Jacobi equation for travel time (eikonal equation), and the Hamiltonian formulation is thus more natural.

(d) A non-convex figuratrix, often appearing in the high-frequency approximation of wave propagation, corresponds to the multivalued Finslerian metric function with the multivalued indicatrix.

Figuratrix:



Indicatrix:



(e) For some problems (e.g., non-scalar waves in anisotropic media, electrons in an electromagnetic field, sound waves in flowing media), a very simple Hamiltonian function may generate the very intricate Finslerian metric function.

Definition of the Finslerian metric function
in terms of the Hamiltonian function:

For $\forall y_i$ satisfying $H(x^m, y_n) = C$ and $\forall\, t \geq 0$:

$$F\left(x^m, t\,\frac{\partial H}{\partial y_n}\right) = t\,\frac{\partial H}{\partial y_k}\,y_k$$

(f) The Hamiltonian formulation allows for very general Hamiltonian functions. The same Finsler space may correspond to various Hamiltonian functions, which provides us with many new possibilities of optimizing perturbation expansions when working with multiparametric families of Finsler spaces.

Examples of different Hamiltonian functions
yielding coinciding geodesics and equal Characteristic functions
but different perturbation expansions:

$$H(x^m, y_n) = N^{-1}\left[y_i\, g^{ij}(x^m, y_n)\, y_j\right]^{\frac{N}{2}}$$

$$H(x^m, y_n) = N^{-1}(y_k y_k)^{\frac{N}{2}}\left\{1 - \left[y_i\, g^{ij}(x^m, y_n)\, y_j\right]^{-\frac{N}{2}}\right\}$$

(g) The Hamiltonian formulation is applicable at the light cone of the space-time ray theory.

The characteristic function is defined only along the geodesics forming light cones and is zero there.

The Finslerian metric function is defined only at the light cone in the cotangent space and is thus not Fréchet-differentiable.

The Finslerian metric function is zero at the light cone in the cotangent space.

The relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function

The relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a Hamiltonian function homogeneous of the second order with respect to the slowness vector,
which corresponds to the Finslerian formulation,
was presented at the 4^{th} International Conference "Finsler Extensions of Relativity Theory" in Cairo, Egypt, 2008.

Here we present
the relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a considerably general Hamiltonian function.

Characteristic function

(point-to-point distance, two-point travel time)

The Characteristic function from point $\tilde{x}^n$ to point x^m:

$$V(x^m, \tilde{x}^n) \quad .$$

The characteristic function satisfies the Hamilton-Jacobi equations

$$H\Big(x^m, \tfrac{\partial V}{\partial x^n}(x^a, \tilde{x}^b)\Big) = C$$

and

$$H\Big(\tilde{x}^m, -\tfrac{\partial V}{\partial \tilde{x}^n}(x^a, \tilde{x}^b)\Big) = C \quad .$$

Equations of geodesics

Hamilton's equations
(equations of geodesics, equations of rays, ray tracing equations):

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\partial H}{\partial y_i}(x^m, y_n) \quad ,$$

$$\frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial H}{\partial x^i}(x^m, y_n) \quad .$$

Hamilton's equations define function

$$t(x^m, \tilde{x}^n)$$

from point $\tilde{x}^n$ to point x^m, with initial conditions $t(\tilde{x}^m, \tilde{x}^n) = 0$.

Propagator matrix of geodesic deviation

The propagator matrix of geodesic deviation from point $\tilde{x}^b$ to point x^a:

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \dfrac{\partial x^i}{\partial \tilde{x}^j} & \dfrac{\partial x^i}{\partial \tilde{y}_j} \\[2ex] \dfrac{\partial y_i}{\partial \tilde{x}^j} & \dfrac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad ,$$

where the derivatives of final point x^i and final slowness vector y_i with respect to initial point $\tilde{x}^j$ and initial slowness vector $\tilde{y}_j$ are taken at fixed parameter t along geodesics (rays).

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a homogeneous Hamiltonian function of the second degree

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial x^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^i}\frac{\partial V}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{V}\frac{\partial V}{\partial \tilde{x}^j}\frac{\partial V}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad .$$

Kronecker delta δ_i^k represents the components of the identity matrix.

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a considerably general Hamiltonian function

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\theta} \frac{\partial t}{\partial x^i} \frac{\partial t}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\theta} \frac{\partial t}{\partial \tilde{x}^i} \frac{\partial t}{\partial x^j} \right) \frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j} \left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\theta} \frac{\partial t}{\partial \tilde{x}^j} \frac{\partial t}{\partial \tilde{x}^k} \right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad ,$$

where integral

$$\theta = \int_0^t \left(\frac{\partial t}{\partial x^r} \frac{\partial^2 H}{\partial y_r \partial y_s} \frac{\partial t}{\partial x^s} \right) \mathrm{d}t$$

is calculated along the geodesic. Kronecker delta δ_i^k represents the components of the identity matrix.

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a considerably general Hamiltonian function

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\theta}\frac{\partial t}{\partial x^i}\frac{\partial t}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\theta}\frac{\partial t}{\partial \tilde{x}^i}\frac{\partial t}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\theta}\frac{\partial t}{\partial \tilde{x}^j}\frac{\partial t}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad .$$

The identities are not applicable if integral θ is equal to zero, which may happen, e.g., if Hamiltonian function $H(x^i, y_j)$ is a homogeneous function of the first degree with respect to y_n.

The identities are applicable to the spatial ray methods but not to the space-time ray methods, because the characteristic function
is not Fréchet-differentiable in the space-time ray methods.

Relation between the propagator matrix of geodesic deviation and the second-order derivatives of the characteristic function for a considerably general Hamiltonian function

$$\left(\frac{\partial^2 V}{\partial x^i \partial x^j} + \frac{1}{\theta}\frac{\partial t}{\partial x^i}\frac{\partial t}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = \frac{\partial y_i}{\partial \tilde{y}_k} \quad ,$$

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\theta}\frac{\partial t}{\partial \tilde{x}^i}\frac{\partial t}{\partial x^j}\right)\frac{\partial x^j}{\partial \tilde{y}_k} = -\delta_i^k \quad ,$$

$$\frac{\partial x^i}{\partial \tilde{y}_j}\left(\frac{\partial^2 V}{\partial \tilde{x}^j \partial \tilde{x}^k} + \frac{1}{\theta}\frac{\partial t}{\partial \tilde{x}^j}\frac{\partial t}{\partial \tilde{x}^k}\right) = \frac{\partial x^i}{\partial \tilde{x}^k} \quad ,$$

where integral

$$\theta = \int_0^t \left(\frac{\partial t}{\partial x^r}\frac{\partial^2 H}{\partial y_r \partial y_s}\frac{\partial t}{\partial x^s}\right)\mathrm{d}t$$

is calculated along the geodesic. Kronecker delta δ_i^k represents the components of the identity matrix.

How to calculate $\partial t/\partial x^i$, $\partial t/\partial \tilde{x}^i$ and θ ?

Information on $\partial t/\partial x^i$, $\partial t/\partial \tilde{x}^i$ and θ is contained in the propagator matrix of geodesic deviation:

$$\mathbf{\Pi}(x^a, \tilde{x}^b) = \begin{pmatrix} \dfrac{\partial x^i}{\partial \tilde{x}^j} & \dfrac{\partial x^i}{\partial \tilde{y}_j} \\ \dfrac{\partial y_i}{\partial \tilde{x}^j} & \dfrac{\partial y_i}{\partial \tilde{y}_j} \end{pmatrix} \quad .$$

Hamilton's (1837) identities

$$\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} \frac{\partial H}{\partial y_j} = 0 \quad ,$$

$$\frac{\partial H}{\partial \tilde{y}_i} \frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} = 0 \quad .$$

Denote

$$X^{i\tilde{a}} = \frac{\partial x^i}{\partial \tilde{y}_a} \quad .$$

Matrix $X_{\tilde{a}k}$ inverse to matrix $X^{i\tilde{a}}$:

$$X^{i\tilde{a}} X_{\tilde{a}k} = \delta^i_k \quad .$$

The second identity:

$$\left(\frac{\partial^2 V}{\partial \tilde{x}^i \partial x^j} + \frac{1}{\theta} \frac{\partial t}{\partial \tilde{x}^i} \frac{\partial t}{\partial x^j} \right) = -X_{\tilde{i}j} \quad .$$

Multiplication by $\partial H / \partial \tilde{y}_i$ yields

$$\frac{\partial t}{\partial x^j} = \theta \, \frac{\partial H}{\partial \tilde{y}_i} \, X_{\tilde{i}j} \quad .$$

Multiplication by $\partial H / \partial y_j$ yields

$$\frac{\partial t}{\partial \tilde{x}^i} = -\theta \, X_{\tilde{i}j} \, \frac{\partial H}{\partial y_j} \quad .$$

Multiplication by both $\partial H / \partial \tilde{y}_i$ and $\partial H / \partial y_j$ yields

$$\theta = \left(\frac{\partial H}{\partial \tilde{y}_i} \, X_{\tilde{i}j} \, \frac{\partial H}{\partial y_j} \right)^{-1} \quad .$$

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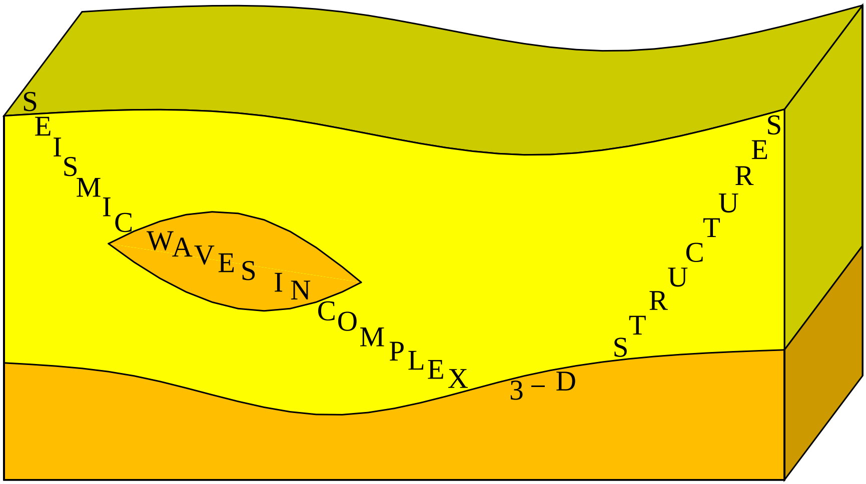

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