Construction of finite-difference schemes in the vicinity of curved interfaces

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Elastodynamic equation

Velocity model: $c_{ijkl} = c_{ijkl}(\mathbf{x})$, $\varrho = \varrho(\mathbf{x})$

For the second-order finite differences with respect to time, we need the the second partial time derivative of the wavefield, which is given by the elastodynamic equation

$$\ddot{u}_i = \varrho^{-1}(c_{ijkl}u_{k,lj} + c_{ijkl,j}u_{k,l})$$

Wavefield $u_k = u_k(\mathbf{x})$ rapidly oscillates, and spatial derivatives $u_{k,lj}$ and $u_{k,l}$ have to be obtained by finite differences in space.

Outside interfaces: c_{ijkl} are smooth, and $c_{ijkl,j}$ can be obtained by finite differences in space.

Vicinity of interfaces: c_{ijkl} and $c_{ijkl,j}$ are known.

Simplifications for this presentation:

Second-order finite differences with respect to time

~~Rectangular FD grid~~

No staggered finite-difference schemes

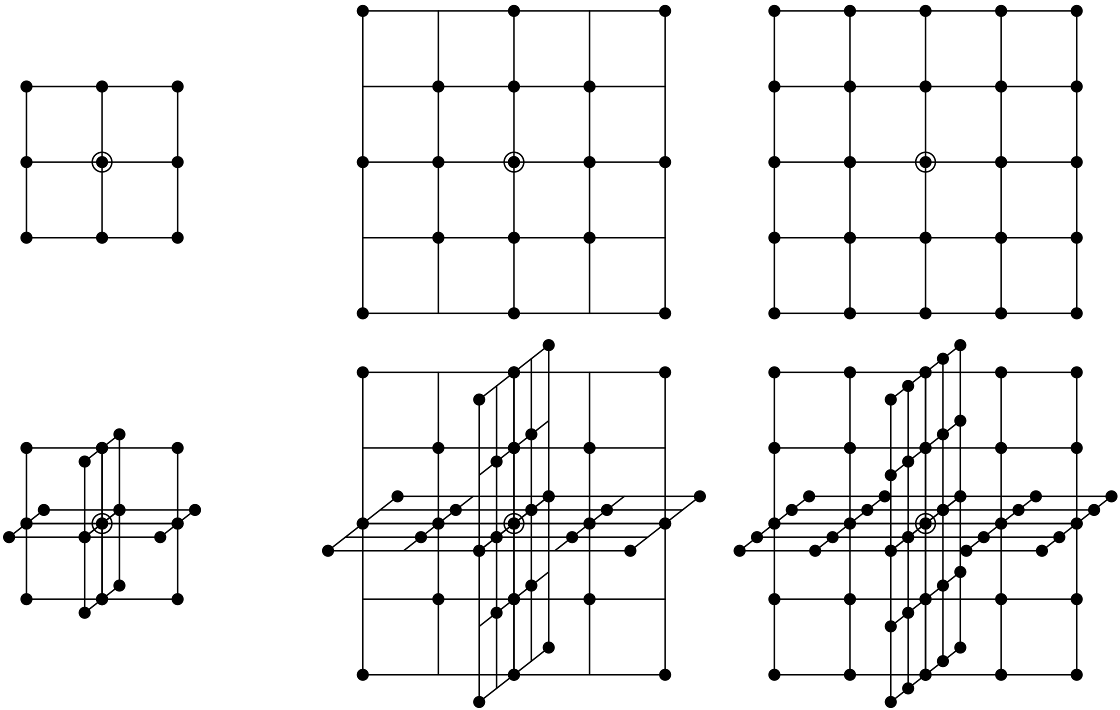
Regular stiffness matrix (no liquid media)

Radii of the interface curvature larger than wavelengths

Parametric description of interfaces

Examples of finite-difference schemes

A finite-difference scheme is determined by the set of points from which the wavefield derivatives at a central point are calculated:



Vector of finite-difference scheme wavefield values

Grid intervals in a rectangular grid:

$$\mathbf{h}_1 = (h, 0, 0)^{\mathrm{T}} \ , \quad \mathbf{h}_2 = (0, h, 0)^{\mathrm{T}} \ , \quad \mathbf{h}_3 = (0, 0, h)^{\mathrm{T}}$$

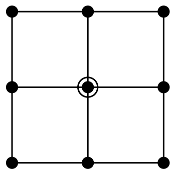
Grid intervals in a triangular or tetrahedral grid:

$$\mathbf{h}_1 = (h, 0, 0)^{\mathrm{T}} \ , \quad \mathbf{h}_2 = (\tfrac{1}{2}h, \tfrac{\sqrt{3}}{2}h, 0)^{\mathrm{T}} \ , \quad \mathbf{h}_3 = (\tfrac{1}{2}h, \tfrac{1}{2\sqrt{3}}h, \sqrt{\tfrac{2}{3}}h)^{\mathrm{T}}$$

For fixed gridpoint $\mathbf{x}$, we define vector $\mathbf{u}$ composed of the FD-scheme wavefield values

$$u_{i(n_1,n_2,n_3)} = u_i(\mathbf{x} + n_1\mathbf{h}_1 + n_2\mathbf{h}_2 + n_3\mathbf{h}_3)$$

Example:



$$\mathbf{u} = \left(u_{i(-1,-1)}, u_{i(0,-1)}, u_{i(1,-1)}, u_{i(-1,0)}, u_{i(0,0)}, u_{i(1,0)}, u_{i(-1,1)}, u_{i(0,1)}, u_{i(1,1)}\right)^{\mathrm{T}}$$

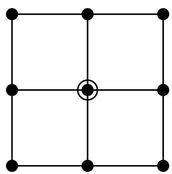
Vector of wavefield derivatives

We denote the value and partial derivatives of the wavefield at $\mathbf{x}$ by

$$
\begin{aligned}
U_i &= u_i(\mathbf{x}) \\
U_{i[j]} &= u_{i,j}(\mathbf{x}) \\
U_{i[jk]} &= u_{i,jk}(\mathbf{x}) + u_{i,kj}(\mathbf{x}) \\
U_{i[jkl]} &= u_{i,jkl}(\mathbf{x}) + u_{i,kjl}(\mathbf{x}) + u_{i,jlk}(\mathbf{x}) + u_{i,klj}(\mathbf{x}) + u_{i,ljk}(\mathbf{x}) + u_{i,lkj}(\mathbf{x}) \\
&\vdots
\end{aligned}
$$

We select the combinations of the partial derivatives which can be determined from wavefield values $\mathbf{u}$, and compose them into vector $\mathbf{U}$.

Example:



$$\mathbf{U} = \left(U_i, U_{i[1]}, U_{i[2]}, U_{i[11]}, U_{i[12]}, U_{i[22]}, U_{i[112]}, U_{i[122]}, U_{i[1122]}\right)^{\mathrm{T}}$$

Elastodynamic equation

$$\mathbf{t} = \mathbf{S}\,\mathbf{U}$$

where

$$\mathbf{t} = (\ddot{u}_1, \ddot{u}_2, \ddot{u}_3)^{\mathrm{T}}$$

The only non–zero elements of matrix $\mathbf{S}$ are

$$S_{n\ \ i[j]} = \varrho^{-1} C_{ijnkk}\ , \quad S_{n\ \ i[jk]} = \varrho^{-1} (C_{ijkn} + C_{ikjn})/2$$

projecting $U_{i[j]}$ and $U_{i[jk]}$ onto $\ddot{u}_n$.

Notation:

$$C_{ijkl} = c_{ijkl}(\mathbf{x})\ , \quad C_{ijklm} = c_{ijkl,m}(\mathbf{x})$$

Taylor expansion

Taylor expansion may be expressed as the linear transformation

$$\mathbf{u} = \mathbf{T}\,\mathbf{U}$$

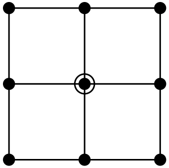
The elements of Taylor–expansion matrix **T**, projecting U_i, $U_{i[j]}$, $U_{i[jk]}$, $U_{i[jkl]}$, $U_{i[jklm]}$, ... onto $u_{n(n_1,n_2,n_3)}$, in a rectangular grid are

$$T_{n(n_1,n_2,n_3):i} = \delta_{ni} \ ,$$
$$T_{n(n_1,n_2,n_3):i[j]} = \delta_{ni} h n_j \ ,$$
$$T_{n(n_1,n_2,n_3):i[jk]} = \delta_{ni} \frac{h^2}{2} n_j n_k \ ,$$
$$T_{n(n_1,n_2,n_3):i[jkl]} = \delta_{ni} \frac{h^3}{6} n_j n_k n_l \ ,$$
$$T_{n(n_1,n_2,n_3):i[jklm]} = \delta_{ni} \frac{h^4}{24} n_j n_k n_l n_m \ , \quad \ldots$$

The combinations representing the first index and the second index of each element of **T** are separated by a colon.

Taylor expansion

Example:

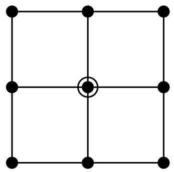


$$
\mathbf{T} = \begin{array}{c} \\ u_{i(\text{-}1,\text{-}1)} \\ u_{i(0,\text{-}1)} \\ u_{i(1,\text{-}1)} \\ u_{i(\text{-}1,0)} \\ u_{i(0,0)} \\ u_{i(1,0)} \\ u_{i(\text{-}1,1)} \\ u_{i(0,1)} \\ u_{i(1,1)} \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} U_i & U_{i[1]} & U_{i[2]} & U_{i[11]} & U_{i[12]} & U_{i[22]} & U_{i[112]} & U_{i[122]} & U_{i[1122]} \end{array} \\
\begin{pmatrix}
1 & -h & -h & \frac{h^2}{2} & \frac{h^2}{2} & \frac{h^2}{2} & -\frac{h^3}{6} & -\frac{h^3}{6} & \frac{h^4}{24} \\
1 & 0 & -h & 0 & 0 & \frac{h^2}{2} & 0 & 0 & 0 \\
1 & h & -h & \frac{h^2}{2} & -\frac{h^2}{2} & \frac{h^2}{2} & -\frac{h^3}{6} & \frac{h^3}{6} & \frac{h^4}{24} \\
1 & -h & 0 & \frac{h^2}{2} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & h & 0 & \frac{h^2}{2} & 0 & 0 & 0 & 0 & 0 \\
1 & -h & h & \frac{h^2}{2} & -\frac{h^2}{2} & \frac{h^2}{2} & \frac{h^3}{6} & -\frac{h^3}{6} & \frac{h^4}{24} \\
1 & 0 & h & 0 & 0 & \frac{h^2}{2} & 0 & 0 & 0 \\
1 & h & h & \frac{h^2}{2} & \frac{h^2}{2} & \frac{h^2}{2} & \frac{h^3}{6} & \frac{h^3}{6} & \frac{h^4}{24}
\end{pmatrix}
\end{array}
$$

Finite-difference scheme:

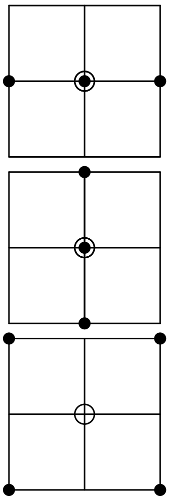
$$\mathbf{t} = \mathbf{S}\,\mathbf{U} \quad , \qquad \mathbf{U} = \mathbf{T}^{-1}\mathbf{u}$$

We need only components $U_{i[j]}$ and $U_{i[jk]}$ of vector $\mathbf{U}$, and combine them into components $\ddot{u}_n$ of vector $\mathbf{t}$.

Example: 

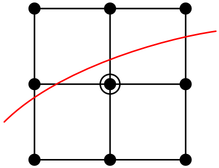
$$\mathbf{T}^{-1} = \begin{matrix} \\ U_{i[1]} \\ U_{i[2]} \\ U_{i[11]} \\ U_{i[12]} \\ U_{i[22]} \end{matrix} \begin{matrix} u_{i(\text{-}1,\text{-}1)} & u_{i(0,\text{-}1)} & u_{i(1,\text{-}1)} & u_{i(\text{-}1,0)} & u_{i(0,0)} & u_{i(1,0)} & u_{i(\text{-}1,1)} & u_{i(0,1)} & u_{i(1,1)} \end{matrix} \atop \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2h} & 0 & \frac{1}{2h} & 0 & 0 & 0 \\ 0 & -\frac{1}{2h} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2h} & 0 \\ 0 & 0 & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & 0 & 0 \\ \frac{1}{2h^2} & 0 & -\frac{1}{2h^2} & 0 & 0 & 0 & -\frac{1}{2h^2} & 0 & \frac{1}{2h^2} \\ 0 & \frac{1}{h^2} & 0 & 0 & -\frac{2}{h^2} & 0 & 0 & \frac{1}{h^2} & 0 \end{pmatrix}$$

Outside interfaces:



Derivatives of u_1, u_2, u_3 separated.

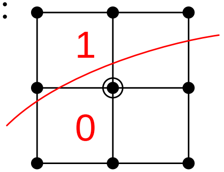
Vicinity of an interface:



Derivatives of u_1, u_2, u_3 mixed.

Filtering matrices according to blocks separated by interfaces:

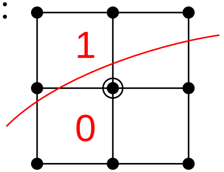
Example:



$$
\mathbf{P}_0 = \begin{array}{c} \\ u_{i(-1,-1)} \\ u_{i(0,-1)} \\ u_{i(1,-1)} \\ u_{i(-1,0)} \\ u_{i(0,0)} \\ u_{i(1,0)} \\ u_{i(-1,1)} \\ u_{i(0,1)} \\ u_{i(1,1)} \end{array}
\begin{array}{c} \begin{array}{ccccccccc} u_{i(-1,-1)} & u_{i(0,-1)} & u_{i(1,-1)} & u_{i(-1,0)} & u_{i(0,0)} & u_{i(1,0)} & u_{i(-1,1)} & u_{i(0,1)} & u_{i(1,1)} \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \end{array}
$$

Filtering matrices according to blocks separated by interfaces:

Example:



$$
\mathbf{P}_1 = \begin{array}{c} \\ u_{i(\text{-}1,\text{-}1)} \\ u_{i(0,\text{-}1)} \\ u_{i(1,\text{-}1)} \\ u_{i(\text{-}1,0)} \\ u_{i(0,0)} \\ u_{i(1,0)} \\ u_{i(\text{-}1,1)} \\ u_{i(0,1)} \\ u_{i(1,1)} \end{array}
\begin{array}{c} \begin{array}{ccccccccc} u_{i(\text{-}1,\text{-}1)} & u_{i(0,\text{-}1)} & u_{i(1,\text{-}1)} & u_{i(\text{-}1,0)} & u_{i(0,0)} & u_{i(1,0)} & u_{i(\text{-}1,1)} & u_{i(0,1)} & u_{i(1,1)} \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}
$$

Taylor expansion:

$$\mathbf{u} = \begin{pmatrix} \mathbf{P}_0\mathbf{T} & \mathbf{P}_1\mathbf{T} \end{pmatrix} \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \end{pmatrix}$$

Insufficient number of equations.

We need boundary conditions between $\mathbf{U}_0$ and $\mathbf{U}_1$ at the interface.

Interfaces

Parametric form of an interface:

$$y_i = y_i(\xi_1, \xi_2)$$

Taylor expansion of $z_i(\xi_1, \xi_2) = y_i(\xi_1, \xi_2) - x_i$:

$$z_i(\xi_1, \xi_2) = Z_i + Z_{iJ}\xi_J + \frac{1}{2}Z_{iJK}\xi_J\xi_K + \frac{1}{6}Z_{iJKL}\xi_J\xi_K\xi_L + ...$$

For radii of the interface curvature larger than wavelengths:

$$z_i(\xi_1, \xi_2) \approx Z_i + Z_{iJ}\xi_J$$

Taylor expansion of the normal to the interface:

$$n_i(\xi_1, \xi_2) = N_i + N_{iJ}\xi_J + \frac{1}{2}N_{iJK}\xi_J\xi_K + \frac{1}{6}N_{iJKL}\xi_J\xi_K\xi_L + ...$$

For radii of the interface curvature larger than wavelengths:

$$n_i(\xi_1, \xi_2) \approx N_i$$

Quantities continuous across the interface:

(a) Wavefield:

$$u_i(\xi_1,\xi_2) = U_i + U_{ij} z_j + \tfrac{1}{2} U_{ijk} z_j z_k + \tfrac{1}{6} U_{ijkl} z_j z_k z_l + \tfrac{1}{24} U_{ijklm} z_j z_k z_l z_m + \ldots$$

(b) Traction $\sigma_{ij} n_j = c_{ijkl} n_j u_{k,l}$:

$$[\sigma_{ij} n_j](\xi_1,\xi_2) \approx C_{ijkl} n_j \big[U_{kl} + U_{klm} z_m + \tfrac{1}{2} U_{klmn} z_m z_n + \tfrac{1}{6} U_{klmnp} z_m z_n z_p + \ldots\big]$$

(c) Second derivative of the wavefield with respect to time:

$$\ddot{u}_i(\xi_1,\xi_2) \approx \varrho^{-1} C_{ijkl} \big[U_{klj} + U_{kljm} z_m + \tfrac{1}{2} U_{kljmn} z_m z_n + \ldots\big]$$

(d) Second derivative of the traction with respect to time:

$$[\ddot{\sigma}_{ij} n_j](\xi_1,\xi_2) = C_{ijkl} n_j \varrho^{-1} \approx \varrho^{-1} C_{ijkl} n_j C_{kmnp} \big[U_{npml} + U_{npmlq} z_q + \ldots\big]$$

(e) Fourth derivative of the wavefield with respect to time

$$\ddot{\ddot{u}}_i(\xi_1,\xi_2) \approx \varrho^{-2} C_{ijkl} C_{kmnp} \big[U_{npmlj} + \ldots\big]$$

...

We insert Taylor expansions

$$z_i(\xi_1, \xi_2) \approx Z_i + Z_{iJ}\xi_J$$

and

$$n_i(\xi_1, \xi_2) \approx N_i$$

(a) Wavefield

$$
\begin{aligned}
u_i(\xi_1,\xi_2) \approx\ & (U_i + U_{ij}Z_j + \tfrac{1}{2}U_{ijk}Z_jZ_k + \tfrac{1}{6}U_{ijkl}Z_jZ_kZ_l + ...) \\
& + (U_{ij}Z_{jJ} + U_{ijk}Z_{jJ}Z_k + \tfrac{1}{2}U_{ijkl}Z_{jJ}Z_kZ_l + ...)\ \xi_J \\
& + (\tfrac{1}{2}U_{ijk}Z_{jJ}Z_{kK} + \tfrac{1}{2}U_{ijkl}Z_{jJ}Z_{kK}Z_l + ...)\,\xi_J\xi_K \\
& + (\tfrac{1}{6}U_{ijkl}Z_{jJ}Z_{kK}Z_{lL} + ...)\,\xi_J\xi_K\xi_L + ...
\end{aligned}
$$

(b) Traction

$$
\begin{aligned}
[\sigma_{ij}n_j](\xi_1,\xi_2) \approx C_{ijkl}N_j \Big[& (U_{kl} + U_{klm}Z_m + \tfrac{1}{2}U_{klmn}Z_mZ_n + ...) \\
& + (U_{klm}Z_{mJ} + U_{klmn}Z_{mJ}Z_n + ...)\,\xi_J \\
& + (\tfrac{1}{2}U_{klmn}Z_{mJ}Z_{nK} + ...)\,\xi_J\xi_K + ... \Big]
\end{aligned}
$$

(c) Second derivative of the wavefield with respect to time

$$
\ddot{u}_i(\xi_1,\xi_2) \approx \varrho^{-1}C_{ijkl}\Big[(U_{klj} + U_{kljm}Z_m + ...) + (U_{kljm}Z_{mJ} + ...)\,\xi_J + ...\Big]
$$

(d) Second derivative of the traction with respect to time

$$
[\ddot{\sigma}_{ij}n_j](\xi_1,\xi_2) \approx \varrho^{-1}C_{ijkl}N_jC_{kmnp}[U_{npml} + ...]
$$

. . .

We arrange the coefficients of the Taylor expansions

$$
\begin{aligned}
u_n(\xi_1,\xi_2) &= m_n + m_{nJ}\xi_J + \tfrac{1}{2}m_{nJK}\xi_J\xi_K + \tfrac{1}{6}m_{nJKL}\xi_J\xi_K\xi_L \\
&\qquad + \tfrac{1}{24}m_{nJKLM}\xi_J\xi_K\xi_L\xi_M + \ldots \\
[\sigma_{ij}n_j](\xi_1,\xi_2) &= m_{n3} + m_{nJ3}\xi_J + \tfrac{1}{2}m_{nJK3}\xi_J\xi_K + \tfrac{1}{6}m_{nJKL3}\xi_J\xi_K\xi_L \\
&\qquad + \tfrac{1}{24}m_{nJKLM3}\xi_J\xi_K\xi_L\xi_M + \ldots \\
\ddot{u}_n(\xi_1,\xi_2) &= m_{n33} + m_{nJ33}\xi_J + \tfrac{1}{2}m_{nJK33}\xi_J\xi_K + \tfrac{1}{6}m_{nJKL33}\xi_J\xi_K\xi_L + \ldots \\
[\ddot{\sigma}_{ij}n_j](\xi_1,\xi_2) &= m_{n333} + m_{nJ333}\xi_J + \tfrac{1}{2}m_{nJK333}\xi_J\xi_K + \ldots \\
\ddot{\ddot{u}}_n(\xi_1,\xi_2) &= m_{n3333} + m_{nJ3333}\xi_J + \ldots \\
&\vdots
\end{aligned}
$$

of the quantities continuous along the interface into vector $\mathbf{m}$.

These quantities at both sides of the interface are equal if the vectors $\mathbf{m}$ at both sides of the interface are equal.

Coefficients $\mathbf{m}$ linearly depend on vector $\mathbf{U}$ of the wavefield derivatives:

$$\mathbf{m} = \mathbf{M}\mathbf{U}$$

$$\mathbf{M} = \begin{pmatrix} \delta_{ni} & \delta_{ni}Z_j & \frac{1}{2}\delta_{ni}Z_jZ_k & \frac{1}{6}\delta_{ni}Z_jZ_kZ_l & \cdots \\ 0 & \delta_{ni}Z_{jJ} & \delta_{ni}Z_{\bar{j}J}Z_{\bar{k}} & \frac{1}{2}\delta_{ni}Z_{\bar{j}J}Z_{\bar{k}}Z_{\bar{l}} & \cdots \\ 0 & 0 & \frac{1}{2}\delta_{ni}Z_{\bar{j}J}Z_{\bar{k}K} & \frac{1}{2}\delta_{ni}Z_{\bar{j}J}Z_{\bar{k}K}Z_{\bar{l}} & \cdots \\ 0 & 0 & 0 & \frac{1}{6}\delta_{ni}Z_{\bar{j}J}Z_{\bar{k}K}Z_{\bar{l}L} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & C_{ijns}N_s & C_{i\bar{j}ns}N_sZ_{\bar{k}} & \frac{1}{2}C_{i\bar{j}ns}N_sZ_{\bar{k}}Z_{\bar{l}} & \cdots \\ 0 & 0 & C_{i\bar{j}ns}N_sZ_{\bar{k}J} & C_{i\bar{j}ns}N_sZ_{\bar{k}J}Z_{\bar{l}} & \cdots \\ 0 & 0 & 0 & \frac{1}{2}C_{i\bar{j}ns}N_sZ_{\bar{k}J}Z_{\bar{l}K} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \varrho^{-1}C_{i\bar{j}\bar{k}n} & \varrho^{-1}C_{i\bar{j}\bar{k}n}Z_{\bar{l}} & \cdots \\ 0 & 0 & 0 & \varrho^{-1}C_{i\bar{j}\bar{k}n}Z_{\bar{l}J} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & \varrho^{-1}C_{i\bar{j}\bar{k}r}C_{r\bar{l}ns}N_s & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Boundary conditions at the interface:

$$\mathbf{M}_0\mathbf{U}_0 = \mathbf{M}_1\mathbf{U}_1$$

Equations for the finite-difference scheme:

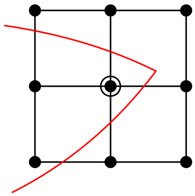
$$\begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_0\mathbf{T} & \mathbf{P}_1\mathbf{T} \\ \mathbf{M}_0 & -\mathbf{M}_1 \end{pmatrix} \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \end{pmatrix}$$

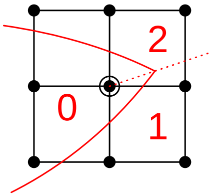
Finite-difference scheme in the vicinity of the interface:

$$\mathbf{t} = \mathbf{S}\,\mathbf{U}_0 \quad , \qquad \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_0\mathbf{T} & \mathbf{P}_1\mathbf{T} \\ \mathbf{M}_0 & -\mathbf{M}_1 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix}$$

Of matrix $\begin{pmatrix} \mathbf{P}_0\mathbf{T} & \mathbf{P}_1\mathbf{T} \\ \mathbf{M}_0 & -\mathbf{M}_1 \end{pmatrix}^{-1}$, we only need the columns corresponding to $\mathbf{u}$ and the rows corresponding to the first and second derivatives contained in $\mathbf{U}_0$. Resulting vector $\mathbf{t}$ has components $\ddot{u}_n$.

Problems with edges and tips





Equations for the finite-difference scheme:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{P}_0\mathbf{T} & \mathbf{P}_1\mathbf{T} & \mathbf{P}_2\mathbf{T} \\ \mathbf{M}_{10} & -\mathbf{M}_1 & \mathbf{0} \\ \mathbf{M}_{20} & \mathbf{0} & -\mathbf{M}_2 \end{pmatrix} \begin{pmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix}$$

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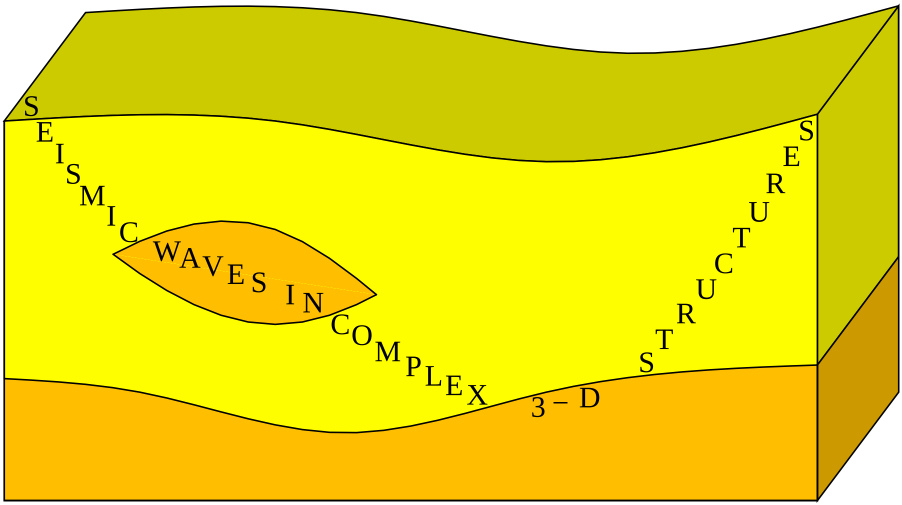

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